REVIEW PROBLEMS FOR MIDTERM 2

(1) What does it mean if $\langle v, w\rangle = 0$?
(2) What is the Cauchy–Schwarz inequality and how does it imply the triangle inequality?
(3) Give upper and lower bounds for $|x|$ in terms of the various $|x_i|$.
(4) If v is a fixed non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ and c is a constant, what does the set of points w with $\langle w, v\rangle = c$ look like?
(5) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be continuous, and prove that f is continuous if and only if each of the component functions $f_1, \ldots, f_m$ is.
(6) If you know that the product functions $p : \mathbb{R}^2 \to \mathbb{R}$ defined by $p(x, y) = xy$ is continuous, how does this imply that a product of two continuous functions is continuous?
(7) Define an open subset of $\mathbb{R}^n$ and a closed subset of $\mathbb{R}^n$.
(8) Prove that $\{x \in \mathbb{R}^n : |x| < 1\}$ is open by hand, and by using the fact that the preimage $f^{-1}(U)$ of an open set U under a continuous map f^{-1} is open.
(9) Must a set be either open or closed? Can it be both? Give examples or proofs as appropriate.
(10) If X is a closed subset of $\mathbb{R}$ with an upper bound, prove that X has a maximum.
(11) What is a compact set? Prove that the following subsets of $\mathbb{R}$ are compact:
  (a) $\{1, 2, 3\}$;
  (b) $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{\frac{-1}{n} : n \in \mathbb{N}\}$;
  (c) $[0, 1]$ (this one is a hard theorem from class).
(12) Write down a closed subset of $\mathbb{R}$ that isn't compact, and a bounded subset of $\mathbb{R}$ that isn't compact. Give proofs.
(13) Prove that the image of a compact set under a continuous function is compact. Is this proof easy or hard?
(14) State the Heine–Borel theorem and use it to prove that a continuous real-valued function on a closed and bounded subset of $\mathbb{R}^n$ has a maximum. What is this theorem called? Is it easy or hard?
(15) Find the matrix of the linear map $\mathbb{R}^3 \to \mathbb{R}^2$ given by $(x, y, z) \mapsto (2x+y, x-y-z)$.
(16) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, let $V = \{x \in \mathbb{R}^n : T(x) = 0\}$. If $v, w \in V$ and $\lambda, \mu \in \mathbb{R}$, prove that $\lambda v + \mu w \in V$. If $n = 3$ what V can you get like this? Give examples.
(17) Find the derivative of $f(x, y) = x^2 y$ at a point (a, b). Write it as a linear map and also in matrix form.
(18) If the height of a hill above sea level is given by $f(x, y) = 2 + x^2 y$ and I am standing on the hill at the point above $(2, 3)$, in which direction should I move to stay at the same height?
(19) Using a previous question and the chain rule, if f and g are functions $\mathbb{R}^n \to \mathbb{R}$ find the derivative of $f^2 g$ at a point (x, y).

(20) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ has derivative $f'(x, y) = (xy \quad x - y)$ in matrix form at a point (x, y). Define $g(u, v) = f(uv, e^{u+v})$. Find $g'(u, v)$.