PRACTISE PROBLEMS FOR MIDTERM 1

1. True/false

(1) If $f_n \to f$ pointwise and each f_n is bounded then f is bounded.
(2) If $f_n \to f$ uniformly and each f_n is continuous then f is continuous.
(3) If $f_n \to f$ uniformly and each f_n is differentiable, then f is differentiable.
(4) $\sum_{n=0}^{\infty} x^n$ converges uniformly to $\frac{1}{1-x}$ on $(-1,1)$.
(5) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly to $\exp(x)$ on $[-1,1]$.
(6) A power series is continuous on $(-R,R)$ if R is its radius of convergence.
(7) If $\sum_{n=0}^{\infty} a_n x^n$ is a power series of radius of convergence R, then it must diverge at $x = R$ or $x = -R$.
(8) If $w, z \in \mathbb{C}$ then $w\overline{z} + \overline{w}z$ is real.
(9) If $v, w, z \in \mathbb{C}$ lie on a straight line then so do v^2, w^2, z^2.
(10) If $w, z \in \mathbb{C}$ then $\arg(wz) = \arg(w) + \arg(z)$.
(11) $\lim_{z\to a} |z| = |a|$ for all a.
(12) $\lim_{z\to a} \arg(z) = \arg(a)$ for all $a \neq 0$.
(13) If α is a complex number then there is a real quadratic $x^2 + ax + b$ with α as a root.
(14) If α, β are complex numbers then there is a real cubic $x^3 + ax^2 + bx + c$ with α and β as roots.
(15) If f is a real polynomial of degree n with n distinct real roots, then every root of f' is real.
(16) If $\exp(w) = \exp(z)$ then $w = z$.
(17) $\exp(\frac{i\pi}{4}) = \frac{1+i}{2}$.
(18) sin is a bounded function on $\mathbb{C}$.
(19) A complex power series of radius of convergence R must diverge at some point z with $|z| = R$.

2. Problems

(1) Suppose that $f_n \to 0$ pointwise and that each f_n is non-negative and decreasing. Prove that $f_n \to 0$ uniformly.
(2) Prove that, if $f_n \to f$ uniformly and g is bounded, then $gf_n \to gf$ uniformly.
(3) Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$ for $x \geq 0$. Prove that

$$\int_0^1 f(x)\,dx = 1.$$

(4) Prove that $1 - x^2 + x^4 - x^6 + \dots$ converges uniformly to $\frac{1}{1+x^2}$ on $(-1,1)$.
(5) Prove that, if $|z| = 1$, then $z + \frac{1}{z}$ is real.
(6) Prove that, if $|v| = |w| = |z| = 1$ and $v + w + z = 0$, then $v^3 = w^3 = z^3$.
(7) Use de Moivre's theorem to prove that

$$\cos(5\theta) = 16\cos^5(\theta) - 20\cos^3(\theta) + 5\cos(\theta).$$

(8) Prove that, if a and b are complex numbers, then there are complex numbers α, β and γ such that $\alpha + \beta + \gamma = a$, $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = b$ and $\alpha\beta\gamma = 1$. *Hint: consider* $z^3 - az^2 + bz - 1$.

(9) If $z \neq 0$ then a **logarithm** of z is any number $L \in \mathbb{C}$ such that $\exp(L) = z$. If $\alpha \in \mathbb{C}$ then a **possible value** of z^α is any number of the form $\exp(\alpha L)$ where L is a logarithm of α

  (a) Find all the logarithms of (-1).

  (b) How many possible values are there of $(-1)^3$, $(-1)^{1/4}$ and $(-1)^{\sqrt{2}}$?