Math 162 – Practise midterm 1

Instructor: Jack Shotton.

Whenever.

Time available: 50 minutes.

This exam is marked out of 40, and counts for 20% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. Let f be a bounded function on $[a, b]$.

   (a) (5 points) Prove that, if $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $[a, b]$, then

   $$L(f, \mathcal{P}) \leq U(f, \mathcal{Q}).$$

   (b) (5 points) If $|f(x)| \leq M$ for all $x \in [a, b]$, prove that

   $$\left| \int_a^b f \right| \leq (b - a)M$$

   .

2. (10 points) (a) (4 points) Prove from the definition that the function $f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x \neq 1 \end{cases}$ is integrable on $[0, 2]$.

   (b) (6 points) Let f be an increasing function on $[0, 1]$. If $\mathcal{P}$ is the partition $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, prove that

   $$U(f, \mathcal{P}) - L(f, \mathcal{P}) = \frac{f(1) - f(0)}{n}.$$

   Deduce that f is integrable.

3. (10 points) State and prove the first fundamental theorem of calculus.

4. (a) (5 points) Define the function cos, and prove that $\cos'(x) = -\sqrt{1 - \cos(x)^2}$.

   (b) (5 points) Let $L(x) = \int_{1/2}^{x} \frac{\log(1-t)}{t}\, dt$ for $x \in (0, 1)$. Prove that

   $$L(x) + L(1 - x) = \log(x) \log(1 - x) - \log(2)^2.$$