Math 162 – Practice final

Instructor: Jack Shotton.

March 6th 2017.

Time available: 120 minutes.

This exam would be marked out of 120, and counts for 50% of the course grade.

Write neatly. Start with the questions you know how to do. If you get stuck for more than a few minutes, move on to another question!

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. (a) (8 points) Prove from the definition that, if f and g are integrable functions on $[a, b]$, then $f + g$ is integrable.

   (b) (4 points) Give (without proof) an example of a function f on $[0, 1]$ which is not integrable.

   (c) (8 points) Suppose that f is a non-negative function on $[0, 1]$ such that, for every $\epsilon > 0$, the number of points x for which $f(x) > \epsilon$ is finite. Prove that f is integrable.

2. (a) (10 points) Prove that, if f is integrable on $[a, b]$ and F is a continuous function on $[a, b]$ such that $F' = f$ on (a, b), then

$$\int_a^b f = F(b) - F(a).$$

   Five points are available if you give a proof assuming that f is continuous.

   (b) (10 points) Suppose that f is a continuous function on $\mathbb{R}_{\geq 0}$ such that $f(x) > 0$ for all x and, for all x,

$$\int_0^x f(t)\, dt = f(x)^3 + 1.$$

   Find f.

3. (a) (8 points) Find the length of the graph of $y = x^{3/2}$ between $x = 0$ and $x = 4$.

   (b) (6 points) Calculate $\int_0^1 \frac{e^x}{1+e^x}\, dx$.

   (c) (6 points) Calculate the volume of a sphere of radius r using the formula

$$\pi \int_a^b f(x)^2\, dx$$

   for the volume enclosed by revolving the graph of $y = f(x)$ between a and b about the x-axis.

4. (a) (15 points) Starting from the fact that a bounded increasing sequence converges, state and prove the Bolzano–Weierstrass theorem.

   (b) (5 points) Write down a sequence $(a_n)_n$ such that, for infinitely many $x \in \mathbb{R}$, $(a_n)_n$ has a subsequence converging to x. *You don't necessarily have to write a formula for your sequence if the pattern/construction is clear.*

5. (a) (8 points) State and prove the alternating series test.

   (b) (12 points) For each of the following series, say whether or not it converges, and give a brief proof:

      1. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$;
      2. $\sum_{n=1}^{\infty} \frac{1}{n \log(n)}$;
      3. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$;

6. (a) (8 points) Prove by integrating $\frac{1}{1+x^2}$ that, for $n \geq 1$, and x a real number

$$\arctan(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + R_n(x)$$

   where $|R_n(x)| \leq \frac{|x|^{2n+3}}{2n+3}$.

   (b) (6 points) Prove that

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{k=0}^{\infty} (-1)^k \frac{1}{3^k(2k+1)}.$$

   (c) (6 points) Prove that $R_n(x)$ is equal to zero to order $2n+2$ at 0. By using a fact about Taylor polynomials (which you should state), find $\arctan^{(5)}(0)$.