Math 163 section 21 – practice final

Instructor: Jack Shotton.

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Time available: 2 hours.

This exam is marked out of 120, and counts for 50% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. (a) (8 points) State and prove the Weierstrass M-test.

   (b) (4 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}.$$

   Decide, with proof, whether this series converges uniformly on $\mathbb{R}$.

   (c) (4 points) Citing an appropriate theorem from class, show that, for every $m \in \mathbb{Z}$,

$$\int_0^{2\pi} f(x) \sin(mx) dx$$

   exists and find its value.

   You may assume that

$$\int_0^{2\pi} \sin(mx) \sin(nx) dx = \begin{cases} 0 & \textit{if } m \neq n \\ \pi & \textit{if } m = n. \end{cases}$$

   (d) (4 points) Carefully citing an appropriate theorem from class, show that f is everywhere differentiable and find $f'(0)$ as an infinite sum.

2. Let $w = \cos(\theta) + i \sin(\theta)$ for some $\theta \in \mathbb{R}$.

   (a) (2 points) Prove that $|w| = 1$ and that $w^{-1} = \overline{w}$.

   (b) (6 points) Prove by induction on n that, if $w = \cos(\theta) + i \sin(\theta)$, then

$$w^n = \cos(n\theta) + i \sin(n\theta)$$

   for all $n \geq 0$.

   (c) (4 points) Find an expression for $\cos(n\theta)$ in terms of w and $\overline{w}$.

   (d) (8 points) Show that

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \frac{4 - 2\cos(\theta)}{5 - 4\cos(\theta)}.$$

3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. If $X \subset \mathbb{R}^n$ then

$$f(X) = \{f(x) : x \in X\}.$$

(a) Give examples to show that the following statements are false:

i. (2 points) if X is closed, then $f(X)$ is closed;

ii. (2 points) if X is bounded, then $f(X)$ is bounded.

(b) (8 points) Suppose that $X \subset \mathbb{R}^n$ is compact and that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Prove that the image

$$f(X) = \{f(x) : x \in X\}$$

is compact.

(c) (8 points) Suppose that $X \subset \mathbb{R}^n$ is a compact set and that $g : \mathbb{R}^n \to \mathbb{R}$ is a *(not necessarily continuous)* function with the following property: for every $x \in X$ there is $\epsilon > 0$ such that g is bounded on the set

$$\{y \in X : |y - x| < \epsilon\}.$$

Prove that g is bounded on X.

4. (a) (7 points) Prove from the definition that, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $DT(x) = T$ for all $x \in \mathbb{R}^n$.

(b) (5 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial of the form

$$f(x, y) = \sum_{n=0}^{K} a_n x^n y^{K-n}$$

for some $K \in \mathbb{N}$.

Prove that

$$Kf(x, y) = D_1 f(x, y) + D_2 f(x, y).$$

(c) (4 points) Prove that, if $Df(x, y)$ is zero for some $(x, y) \in \mathbb{R}^2$, then $f(x, y) = 0$.

(d) (4 points) Show that there is no such polynomial f for which $D_1 f(x, y) = y^2$ and $D_2 f(x, y) = x^2$.

5. (a) (10 points) Find, from the definition,

$$\int_{[0,1]^2} xy dx dy.$$

(b) (10 points) Prove that, if $X \subset \mathbb{R}^n$ is bounded and has content zero and A is a closed rectangle containing X, then

$$\int_A 1_X = 0.$$

Remember that 1_X is the indicator function defined by

$$1_X(x) = \begin{cases} 1 & \textit{if } x \in X \\ 0 & \textit{if } x \notin X. \end{cases}$$

6. Let $X \subset \mathbb{R}^2$ be closed, bounded subset of a closed rectangle $A = [a,b]^2$ such that the indicator function 1_X is integrable and satisfies the hypotheses of Fubini's theorem. Recall that $\text{vol}(X) = \int_A 1_X$.

(a) (2 points) If $\lambda > 0$, let $\lambda X = \{\lambda x : x \in X\}$. Show that

$$1_{\lambda X}(x, y) = 1_X(\lambda^{-1}x, \lambda^{-1}y).$$

(b) (8 points) Use Fubini's theorem and two substitutions to show that

$$\text{vol}(\lambda X) = \lambda^2 \text{vol}(X).$$

(c) (10 points) Let $C \subset \mathbb{R}^3$ be the set

$$\{(x, y, z) : 0 \leq z \leq h, (x, y) \in \left(1 - \frac{z}{h}\right) X\}.$$

It is a cone of height h whose base is X.
Prove that $\text{vol}(C) = \frac{1}{3} h \text{vol}(X)$.