COMPLEX POLYNOMIALS

Definition 0.1. A **polynomial** is a function on $\mathbb{C}$ of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

for $a_0, a_1, \ldots, a_n$ complex numbers.

If $a_0, \ldots, a_n \in \mathbb{R}$, then f is a **real** polynomial.

If $a_n \neq 0$, then n is the **degree** of f. We usually just divide by a_n and assume that the first coefficient is 1, so

$$f(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0.$$

In this case we say that f is **monic**.

Theorem 0.2. *(Remainder Theorem) If f is a polynomial of degree n and $\alpha \in \mathbb{C}$, then we can write*

$$f(z) = (z - \alpha) g(z) + r$$

where $r = f(\alpha)$ and g has degree $n - 1$.

Proof. Expanding $z^m = ((z - \alpha) + \alpha)^m$ with the binomial theorem shows that z^m can be written as a polynomial in $(z - \alpha)$, and so $f(z)$ can also be written as a polynomial in $(z - \alpha)$. That is, we can write

$$f(z) = b_n (z - \alpha)^n + \ldots + b_1 (z - \alpha) + b_0.$$

Plugging in $z = \alpha$ shows that $b_0 = f(\alpha)$. Since f has degree n, $b_n \neq 0$. Letting

$$g(z) = b_n (z - \alpha)^{n-1} + \ldots + b_2 (z - \alpha) + b_1,$$

we have $f(z) = (z - \alpha) g(z) + b_0$, where the degree of g is $n - 1$ and $b_0 = f(\alpha)$. □

Corollary 0.3. *(Factor Theorem) If f is a polynomial of degree n and $\alpha \in \mathbb{C}$ satisfies $f(\alpha) = 0$, then we can write*

$$f(z) = (z - \alpha) g(z)$$

where the degree of g is $n - 1$.

Proof. Immediate from the remainder theorem (as $r = 0$). □

In the previous theorems, if f is a real polynomial and α is real, then g and r will also be real (this is easy to see from the proofs).

Over the complex numbers, every polynomial has a root and so every polynomial will factorise into linear factors:

Theorem 0.4. *(Fundamental Theorem of Algebra) If f is a complex polynomial of degree at least one, then there is $\alpha \in \mathbb{C}$ such that $f(\alpha) = 0$.*

We proved this in class.

Corollary 0.5. *If f is a polynomial of degree n, then we can write*

$$f(z) = c(z - \alpha_1)(z - \alpha_2)(\dots)(z - \alpha_n)$$

for $c, \alpha_1, \dots, \alpha_n \in \mathbb{C}$, $c \neq 0$. In other words, f can be factorised into linear factors.

Moreover, the $\alpha_1, \dots, \alpha_n$ are unique up to reordering.

Proof. First we prove that the factorisation exists, by induction: the idea is simply to keep taking out factors using the FTA and the factor theorem.

The base case is $n = 0$, in which case we're just saying that $f(z) = a_0$, which is clear ($c = a_0$ and there are no αs).

Suppose that $n \geq 1$, that every polynomial of degree $n - 1$ can be factored into linear terms, and that f is a polynomial of degree n. Then by the fundamental theorem of algebra, there is $\alpha_n \in \mathbb{C}$ such that $f(\alpha_n) = 0$. By the factor theorem, we can write

$$f(z) = (z - \alpha_n)g(z)$$

for some g of degree $n - 1$. By the induction hypothesis, we can write

$$g(z) = c(z - \alpha_1)(\dots)(z - \alpha_{n-1})$$

for some $c, \alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$. Then

$$\begin{aligned} f(z) &= g(z) \cdots (z - \alpha_n) \\ &= c(z - \alpha_1) \dots (z - \alpha_{n-1})(z - \alpha_n) \end{aligned}$$

as required.

For the uniqueness we also use induction. (Don't worry too much about this part of the proof if it is hard to understand — I think even understanding precisely what the statement means is not so easy.) Clearly c is unique (it is just the coefficient of z^n) so we may divide it out and suppose $c = 1$. We will prove the following statement:

if

$$(z - \alpha_1)(z - \alpha_2)(\dots)(z - \alpha_n) = (z - \alpha'_1)(z - \alpha'_2)(\dots)(z - \alpha'_n)$$

then $(\alpha'_1, \dots, \alpha'_n)$ is a rearrangement of $(\alpha_1, \dots, \alpha_n)$.

For $n = 0$ this is clear. So suppose that $n > 0$ and that the statement is true for $n - 1$. Since α_n is a root of the left hand side, it is also a root of the right hand side, and so we have that

$$(\alpha_n - \alpha'_1)(\dots)(\alpha_n - \alpha'_n)$$

which is only possible if one of the α'_i is actually equal to α_n. Switching places of α'_i and α'_n, we can assume that $\alpha_n = \alpha'_n$. Dividing out $z - \alpha_n$ from both sides gives that

$$(z - \alpha_1)(z - \alpha_2)(\dots)(z - \alpha_{n-1}) = (z - \alpha'_1)(z - \alpha'_2)(\dots)(z - \alpha'_{n-1}).$$

By the induction hypothesis, $(\alpha'_1, \dots, \alpha'_{n-1})$ is a rearrangement of $(\alpha_1, \dots, \alpha_{n-1})$. As $\alpha'_n = \alpha_n$, $(\alpha'_1, \dots, \alpha'_n)$ is a rearrangement of $(\alpha_1, \dots, \alpha_n)$ as required. □

What about real polynomials? They can't always be factored into linear factors over $\mathbb{R}$ — for example, $x^2 + 1$ — but it turns out that they *can* always be factored into linear and quadratic factors over $\mathbb{R}$.

Corollary 0.6. *Let f be a real polynomial of degree n. Then we can write*

$$f(x) = c(x - \alpha_1) \dots (x - \alpha_r) f_1(x) \dots f_s(x)$$

where $c \neq 0$ is real, r and s are non-negative integers with $r + 2s = n$, $\alpha_1, \dots, \alpha_r$ are real numbers, and $f_1, \dots, f_s$ are real quadratics of the form $x^2 - a_i x + b_i$ for $a_i, b_i \in \mathbb{R}$.

Moreover, this way of writing f is unique up to rearrangement.

Proof. We won't bother with the uniqueness (it's similar to before) — we'll just show that the factorisation exists. The basic idea that every root is either real or has to appear together with its complex conjugate — but $(z - \alpha)(z - \overline{\alpha})$ is always a real quadratic. It's easiest to make this precise by induction.

First, note that if $f(z) = a_n z^n + \dots + a_0$ is any complex polynomial, then

$$\begin{aligned} \overline{f(z)} &= \overline{a_n z^n + \dots + a_0} \\ &= \overline{a_n}\, \overline{z}^n + \dots + \overline{a_0}. \end{aligned}$$

If f has real coefficients, then each $\overline{a_i} = a_i$ and so

$$\begin{aligned} \overline{f(z)} &= a_n \overline{z}^n + \dots + a_0 \\ &= f(\overline{z}). \end{aligned}$$

In particular, if $f(\alpha) = 0$ then $f(\overline{\alpha}) = 0$ as well.

We prove by induction on the degree n that f can be factored into real linear and quadratic terms. This is obvious if $n = 0, 1, 2$. So suppose that f has degree $n > 2$ and that every real polynomial of degree $< n$ can be factored into real linear and quadratic terms.

By the fundamental theorem of algebra, f has a root $\alpha \in \mathbb{C}$.

If α is real, then we can write $f(x) = (x - \alpha) g(x)$ with g a real polynomial of degree $n - 1$. Since g can be factored into linear and quadratic terms, so can f.

Otherwise, α is not real and so $\overline{\alpha} \neq \alpha$. By what we said above, $\overline{\alpha}$ is also a root of f and so by the factor theorem we can write

$$\begin{aligned} f(x) &= (x - \alpha)(x - \overline{\alpha}) g(x) \\ &= \left(x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha}\right) g(x) \end{aligned}$$

for some complex polynomial g of degree $n - 2$. But $x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha}$ is a *real* quadratic, because $\alpha + \overline{\alpha}$ and $\alpha\overline{\alpha}$ are always real, and so (by the technical lemma below) g is a real polynomial.

By the induction hypothesis, g can be written as a product of real linear and quadratic factors; as f is g multiplied by a real quadratic, the same is true of f. □

Lemma 0.7. *('technical lemma') If f is a real polynomial, h is a real polynomial, and $f = hg$ with g a complex polynomial, then g is actually a real polynomial.*

Proof. If $P(z) = a_n z^n + \dots a_0$ is any complex polynomial, let $\overline{P}$ be the complex polynomial

$$\overline{a_n} z^n + \dots \overline{a_0}.$$

Then it is easy to see that $\overline{P} = P$ if and only if P is a real polynomial, and that

$$\overline{PQ} = \overline{P} \cdot \overline{Q}$$

for any complex polynomials P and Q. But now

$$hg = f = \overline{f} = \overline{hg} = \overline{h} \cdot \overline{g} = h\overline{g}.$$

So $h(g - \overline{g}) = 0$. But this is only possible if $g - \overline{g} = 0$, so $g = \overline{g}$ and g is real. $\square$