POLYNOMIALS OF ODD DEGREE HAVE A ROOT

The proof of this result I gave in class was a little misleading (although still correct) because it contained an unnecessary step. Here is a better write-up of the proof.

Theorem. *Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0$, and suppose that n is odd. Then there exists $x \in \mathbb{R}$ with $f(x) = 0$.*

Proof. Write $f(x) = x^n g(x)$, so that $g(x) = 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}$. I will choose $M > 0$ sufficiently large that $g(x) > 0$ if $|x| \geq M$ (that I can do this is because $g(x) \to 1$ as $x \to \pm\infty$, but I'll give a direct argument).

Choose M so that $M \geq 1$ and $M \geq 2n|a_i|$ for each $0 \leq i \leq n-1$. Then, if $i \geq 1$ and $|x| \geq M$ then

$$\begin{aligned}
\left|\frac{a_i}{x^i}\right| &\leq \frac{|a_i|}{M^i} \\
&\leq \frac{|a_i|}{M} && \text{as } M \geq 1 \\
&\leq \frac{1}{2n} && \text{as } M \geq 2n|a_i|.
\end{aligned}$$

Therefore, if $|x| \geq M$, then

$$\begin{aligned}
|g(x)| &= \left|1 + \left(\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}\right)\right| \\
&\geq |1| - \left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \dots + \frac{a_0}{x^n}\right| && \text{by } |a-b| \geq |a| - |b| \\
&\geq 1 - \left(\left|\frac{a_{n-1}}{x}\right| + \left|\frac{a_{n-2}}{x^2}\right| + \dots + \left|\frac{a_0}{x^n}\right|\right) && \text{by the triangle inequality} \\
&\geq 1 - \left(\frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n}\right) && \text{with } n \text{ copies of } \tfrac{1}{2n} \\
&= 1 - n \times \frac{1}{2n} \\
&= \frac{1}{2} \\
&> 0.
\end{aligned}$$

Now, $f(M) = M^n g(M) > 0$ as $M > 0$ and $g(M) > 0$, and

$$f(-M) = (-M)^n g(-M) = -M^n g(-M)$$

as n is odd; as $g(-M) > 0$ this shows that $f(-M) < 0$.

Because f is a polynomial, it is continuous on $[-M, M]$. As $f(-M) < 0$ and $f(M) > 0$, by the intermediate value theorem there exists $x \in [-M, M]$ such that $f(x) = 0$. □