Math 162 – Midterm 2

Instructor: Jack Shotton.

February 17th 2017.

Time available: 50 minutes.

This exam is marked out of 40, and counts for 20% of the course grade.

Write neatly. Start with the questions you know how to do. If you get stuck for more than a few minutes, move on to another question!

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. Let f and g be twice-differentiable functions on $\mathbb{R}$.
   (a) (5 points) Suppose that $f''(x) + f(x) = 0$ for all x, and that $f(0) = f'(0) = 0$. Prove that $f(x) = 0$ for all x.
   (b) (5 points) Suppose that $g''(x) + 2g'(x) + g(x) = 0$ for all x. By considering $e^x g(x)$, show that $g(x) = (Ax + B)e^{-x}$ for some constants A and B.

2. Find each of the following indefinite integrals.
   (a) (2½ points)
   $$\int \frac{x+1}{x^2+1}\,dx.$$
   (b) (2½ points)
   $$\int \sqrt{1-x^2}\,dx.$$
   (c) (2½ points)
   $$\int \frac{1}{x^3-x}\,dx.$$
   (d) (2½ points)
   $$\int \frac{\log(\log(x))}{x}\,dx.$$

3. If m, n are non-negative integers, define
   $$B(m,n) = \int_0^1 x^m(1-x)^n\,dx.$$
   (a) (2 points) Write down $B(m,0)$ and $B(0,n)$.
   (b) (4 points) Prove that, if $m > 0$, then
   $$B(m,n) = \frac{m}{n+1}B(m-1,n+1).$$
   (c) (4 points) Prove by induction on m that $B(m,n) = \frac{m!n!}{(m+n+1)!}$. *(remember, $0! = 1$.)*

4. Consider the curve C given parametrically by $(u(t), v(t))$ where $u(t) = \frac{\cos(t)}{t^2}$ and $v(t) = \frac{\sin(t)}{t^2}$ for $t > 0$.
   (a) (4 points) If $X > 1$, show that the arc length of C for t between 1 and X is given by
   $$\int_1^X \frac{\sqrt{t^2+4}}{t^3}\,dt.$$
   (b) (4 points) Prove carefully that $\int_1^\infty \frac{\sqrt{t^2+4}}{t^3}\,dt$ exists. *Do not try to calculate the integral.*
   (c) (2 points) Show that the arc-length of C for t between 1 and ∞ is the same as the length of the graph $y = x^2$ for x between 0 and 1.

   The formula for the arc-length of a parametric curve is $\int_a^b \sqrt{u'(t)^2 + v'(t)^2}\,dt$.