Math 163 – Midterm 2

Instructor: Jack Shotton.

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Time available: 50 minutes.

This exam is marked out of 40, and counts for 20% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. (a) (1 point) Define the scalar product $\langle v, w \rangle$ of two vectors v and w in $\mathbb{R}^n$.
   (b) (5 points) State and prove the Cauchy–Schwarz inequality.
   (c) (4 points) Let $a = (1, 0, 0, 0)$, $b = (0, 1, 0, 0)$, $c = (0, 0, 1, 0)$ and $d = (0, 0, 0, 1)$ be points in $\mathbb{R}^4$. They are the vertices of a regular tetrahedron.

   Let m be the midpoint of the line joining a to b. Let θ be the angle between the lines joining c to m and joining d to m. Find $\cos(\theta)$.

2. (a) (2 points) Define what it means for $U \subset \mathbb{R}^n$ to be open.
   (b) For each of the following statements give either a brief proof or a counterexample as appropriate:
      i. (2 points) If $U, V \subset \mathbb{R}^n$ are open, so is $U \cap V$.
      ii. (2 points) If $U \subset \mathbb{R}^n$ is open and $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then $f(U)$ is open.
      iii. (2 points) If $X_1, X_2, \ldots \subset \mathbb{R}^n$ are closed, then so is their union $\bigcup_{n=1}^{\infty} X_n$.
      iv. (2 points) If $U \subset \mathbb{R}^m$ is open and $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then $f^{-1}(U)$ is open.

3. (a) (5 points) Prove that a compact set $X \subset \mathbb{R}^n$ is bounded.
   (b) (5 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and define $f : \mathbb{R}^n \setminus \{\underline{0}\} \to \mathbb{R}$ by

$$f(x) = \frac{|T(x)|}{|x|}.$$

   Use the extreme value theorem to prove that f has a maximum value on $\mathbb{R}^n \setminus \{\underline{0}\}$. *(hint: consider f restricted to the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$)*

4. (a) (2 points) Define what it means for $f : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable.
   (b) (3 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(x, y, z) = xyz$. From the definition of the derivative, find $Df(x, y, z)$.
   (c) (3 points) Let $g : \mathbb{R} \to \mathbb{R}^3$ be defined as $g(x) = (\cos(x), \sin(x), x)$. Find the derivative of g. Draw a picture of the image of g.
   (d) (2 points) Using the chain rule, write down the derivative $(f \circ g)'(x)$.