Math 161 section 21 – Midterm 2

Instructor: Jack Shotton.

November 7th 2016.

Time available: 50 minutes.

This exam is marked out of 40, and counts for 20% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. (a) (3 points) Define what it means for a function f to be continuous on $\mathbb{R}$.

Solution: It means that, for every $a \in \mathbb{R}$ and every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all $x \in \mathbb{R}$ such that $|x - a| < \delta$.
Alternatively and equivalently, it means that for all $a \in \mathbb{R}$, $\lim_{x \to a} f(x) = f(a)$.

(b) (5 points) Prove that, if f and g are continuous on $\mathbb{R}$, then so is $f \circ g$.

Solution: Let $a \in \mathbb{R}$ and $\epsilon > 0$. As f is continuous at $g(a)$, there is $\delta_1 > 0$ such that $|f(y) - f(g(a))| < \epsilon$ for all y such that $|y - g(a)| < \delta_1$. As g is continuous at a there is $\delta > 0$ such that $|g(x) - g(a)| < \delta_1$ for all $x \in \mathbb{R}$ such that $|x - a| < \delta$. Therefore, if $|x - a| < \delta$ then $|g(x) - g(a)| < \delta_1$ and so (taking $y = g(x)$) $|f(g(x)) - f(g(a))| < \epsilon$, as required.

(c) (2 points) Give an example of functions f and g on $\mathbb{R}$ such that f **is** continuous on $\mathbb{R}$, g **is not** continuous on $\mathbb{R}$, and $f \circ g$ **is** continuous on $\mathbb{R}$. *You do not have to prove that your example works.*

Solution: Let $f(x) = x^2$, and let $g(x) = 1$ if $x \geq 0$ and -1 if $x < 0$. Then $(f \circ g)(x) = 1$ for all x. So $f \circ g$ is continuous, f is continuous, g is not.

2. (a) (3 points) State the extreme value theorem.

Solution: If f is a continuous function on the closed interval $[a, b]$, then f has a maximum and a minimum on $[a, b]$.
(I would accept just maximum or just minimum.)

(b) (4 points) Suppose that f is a continuous function on $[0, \infty)$ such that $f(0) > 0$ and such that

$$\lim_{x \to \infty} f(x) = 0.$$

Prove that f has a maximum.

Solution: As $f(0) > 0$ and $\lim_{x\to\infty} f(x) = 0$, there is $X \in \mathbb{R}$ such that

$$|f(x) - 0| < f(0)$$

for all $x \geq X$. Let x_0 be a point in $[0, X]$ at which f attains a maximum, which exists by the extreme value theorem. Then I claim that x_0 is a maximum for f on all of $[0, \infty)$. Indeed, if $x \in [0, \infty)$, then either $x \leq X$, in which case $f(x) \leq f(x_0)$ by definition of x_0, or $x > X$ in which case $f(x) < f(0)$ by choice of X. In the first case we are done, and in the second case observe that $0 \in [0, X]$, so $f(0) \leq f(x_0)$ and so $f(x) < f(0) \leq f(x_0)$ as required.

(c) (3 points) Give an example of a continuous, bounded function on $(0, 1]$ that has neither a maximum nor a minimum. *You do not have to prove that your example works.*

Solution: Take $f(x) = (1 - x)\sin(\frac{1}{x})$.

3. (a) (3 points) If $S \subset \mathbb{R}$, define what it means for x to be a least upper bound for S.

Solution: It means that x is an upper bound for S and that any other upper bound for S is at least x. In other words, $x \geq y$ for all $y \in S$, and if $x' \geq y$ for all $y \in S$ then $x' \geq x$.

(b) (3 points) Prove from the least upper bound axiom that $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Solution: The least upper bound axiom states that every non-empty, bounded above subset of $\mathbb{R}$ has a least upper bound.

Suppose that $\mathbb{N}$ is bounded above in $\mathbb{R}$. As $\mathbb{N}$ is non-empty, it has a least upper bound, $x = \sup(\mathbb{N})$. Then $x - 1 < x$ and so is not an upper bound for $\mathbb{N}$, there is $n \in \mathbb{N}$ such that $n > x - 1$. But that $n + 1 > x$. As $n + 1 \in \mathbb{N}$ also, this contradicts the assumption that x is an upper bound for $\mathbb{N}$.

(c) (4 points) If A and B are non-empty subsets of $\mathbb{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$, show that $\sup(A)$ and $\inf(B)$ exist and prove that $\sup(A) \leq \inf(B)$.

Solution: By assumption, A and B are non-empty. As A is bounded above (by any element of B) and B is bounded below (by any element of A), $\sup(A)$ and $\inf(B)$ exist.

If $b \in B$, then b is an upper bound for A. Therefore $b \geq \sup(A)$. This shows that $\sup(A)$ is a lower bound for B, and so $\sup(A) \leq \inf(B)$.

4. (10 points) State and prove the intermediate value theorem.

Solution: If f is a continuous function on $[a, b]$ such that $f(a) < c < f(b)$ for some $c \in \mathbb{R}$, then there exists $x \in [a, b]$ such that $f(x) = c$.

Proof. Replacing f by $f - c$, we may assume that $c = 0$.

Let $S = \{x \in [a, b] : f(x) < 0\}$. Then S is non-empty (as $f(a) < 0$ so $a \in S$) and bounded above (by b). Therefore we may let $x_0 = \sup(S)$. Then $a \leq x_0 \leq b$.

I claim that $f(x_0) = 0$; we are then done. The claim is proved by contradiction; suppose that $f(x_0) \neq 0$. Then either $f(x_0) > 0$ or $f(x_0) < 0$.

Suppose first that $f(x_0) > 0$. Then $x_0 > a$ (as $f(a) < 0$, so $x_0 \neq a$) and so f is left-continuous at x_0. Therefore there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$ for all $x \in (x_0 - \delta, x_0]$. In particular, for $x \in (x_0 - \delta, x_0]$,

$$f(x) > f(x_0) - \frac{f(x_0)}{2} > 0.$$

As $x_0 = \sup(S)$ and $x_0 - \delta < x_0$, there is $x \in S$ such that $x_0 \geq x > x_0 - \delta$. Choose such an x. Then $f(x) < 0$, as $x \in S$, but $f(x) > 0$ as $x \in (x_0 - \delta, x_0]$. Contradiction!

Now suppose that $f(x_0) < 0$. Then $x_0 < b$ (as $f(b) > 0$, so $x_0 \neq b$) and so f is right-continuous at x_0. Therefore there exists $\delta > 0$ such that $|f(x) - f(x_0)| < |f(x_0)|/2$ for all $x \in [x_0, x_0 + \delta)$. In particular, for $x \in [x_0.x_0 + \delta)$,

$$f(x) < f(x_0) + \frac{|f(x_0)|}{2} = \frac{f(x_0)}{2} < 0.$$

Choose such an x with $x > x_0$. Then as x_0 is an upper bound for S, $x \notin S$. So $f(x) \geq 0$. But as $x \in [x_0, x_0 + \delta)$, $f(x) < 0$. Contradiction!