

Math 162 – Midterm 1

Instructor: Jack Shotton.

January 27th 2017.

Time available: 50 minutes.

This exam is marked out of 40, and counts for 20% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: \mathbb{Z} denotes the integers, \mathbb{Q} the rational numbers, \mathbb{R} the real numbers, \mathbb{N} the natural numbers $\{1, 2, 3, \dots\}$.

1. Let f be a bounded function on $[a, b]$.

(a) (4 points) Define the upper sum of f with respect to a partition \mathcal{P} , and the upper integral of f over $[a, b]$.

(b) (6 points) Let $f(x) = x^2$. From the definition of the integral, prove that

$$\int_0^1 f = \frac{1}{3}.$$

You may assume that $\sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1)$, and any facts from class about upper and lower integrals.

2. (10 points) Which of the following functions f on $[0, 1]$ are integrable? You do not have to provide proofs.

(a) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

(b) $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$

(c) $f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(d) $f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(e) $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

3. (a) (6 points) Prove that, if f is integrable on $[a, b]$ and $c \in (a, b)$, then the function

$$F(x) = \int_a^x f$$

is continuous at c . *You may assume general facts about integrals, provided that you state them clearly.*

(b) (4 points) If f is a differentiable function, find the derivative of

$$G(x) = \int_0^{f(x)} f(t) dt,$$

explaining your working carefully.

4. (a) (6 points) Define the function \log , and prove that $\log(xy) = \log(x) + \log(y)$ for $x, y > 0$.

(b) (4 points) Let $C(x) = \int_0^x \cos(t^2) dt$. Compute the derivative of $x C(x)$, and hence find an antiderivative of $C(x)$.