Math 163 – Midterm 1

Instructor: Jack Shotton.

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Time available: 50 minutes.

This exam is marked out of 30, and counts for 20% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. Suppose that f_n is a sequence of continuous functions on $[a, b]$ converging pointwise to a function f on $[a, b]$.
   (a) (5 points) Give an example to show that f need not be bounded, and prove that your example works.
   (b) (5 points) Prove that if $f_n \to f$ uniformly then f is continuous.

2. Let
$$a_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$
for $n \geq 0$ (so $a_0 = 1$).
   (a) (3 points) Find the radius of convergence R of
$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$
   (b) (3 points) Prove that $(n+1)a_{n+1} - 4na_n = 2a_n$ for $n \geq 0$ and hence that
$$(1 - 4x)f'(x) = 2f(x)$$
   for $x \in (-R, R)$.
   (c) (3 points) By differentiating $g(x) = f(x)\sqrt{1 - 4x}$ and using the previous part, prove that
$$f(x) = \frac{1}{\sqrt{1 - 4x}}$$
   in $(-R, R)$.
   (d) (1 point) Does the series defining f converge uniformly on $(-R, R)$?

3. Let $f(z) = z^7 - 1$, and let α be the root of f with smallest non-zero argument.
   (a) (2 points) Draw all the roots of f in the complex plane, labelling α.
   (b) (2 points) Show that
$$1 + \alpha + \alpha^2 + \ldots + \alpha^6 = 0.$$
   (Hint: multiply by $\alpha - 1$.)
   (c) (3 points) Let $\beta = \alpha + \alpha^2 + \alpha^4$. Prove that
$$\beta^2 = -2 - \beta.$$
   (d) (3 points) Deduce that
$$\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = \frac{\sqrt{7}}{2}.$$