Math 161 section 21 – Midterm 1

Instructor: Jack Shotton.

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Time available: 50 minutes.

This exam is marked out of 40, and counts for 20% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. (a) (5 points) Prove from the axioms of arithmetic that, if $a \in \mathbb{R}$, then $0 \cdot a = 0$.

Solution: We have the following equalities:

$$\begin{aligned} 0 \cdot a &= (0+0) \cdot a && \text{by AId} \\ &= 0 \cdot a + 0 \cdot a. && \text{by D} \end{aligned}$$

Adding $-0 \cdot a$ to both sides we obtain

$$0 \cdot a + (-0 \cdot a) = (0 \cdot a + 0 \cdot a) + (-0 \cdot a).$$

By AIn, the left hand side is 0. The right hand side, by AA, is

$$\begin{aligned} 0 \cdot a + (0 \cdot a + (-0 \cdot a)) &= 0 \cdot a + 0 && \text{by AIn} \\ &= 0 \cdot a && \text{by AId.} \end{aligned}$$

Therefore $0 = 0 \cdot a$ as required.

(b) (5 points) Prove from the axioms of arithmetic that $x \in \mathbb{R}$ satisfies $x^2 = x$ if and only if $x = 0$ or $x = 1$.

Solution: There are two directions to address: that $x^2 = x$ if $x = 0$ or $x = 1$ (the 'if' direction), and that if $x^2 = x$ then $x = 0$ or $x = 1$ (the 'only if' direction).

For the first direction, if $x = 0$ then $x^2 = 0 \cdot 0 = 0$ by part a, and if $x = 1$ then $x^2 = 1 \cdot 1 = 1$ by MId.

For the other direction, suppose that $x^2 = x$ and $x \neq 0$. We must prove that $x = 1$. As $x \neq 0$ we may use MIn to obtain $x^{-1} \cdot (x^2) = x^{-1} \cdot x = 1$. But

$$\begin{aligned} x^{-1} \cdot (x^2) &= x^{-1} \cdot (x \cdot x) \\ &= (x^{-1} \cdot x) \cdot x && \text{MA} \\ &= 1 \cdot x && \text{MIn} \\ &= x && \text{MId.} \end{aligned}$$

Therefore $x = x^{-1} \cdot (x^2) = 1$ as required.

2. (a) (5 points) Prove from the order axioms that $x^2 \geq 0$ for all $x \in \mathbb{R}$.

Solution: By O1, either $x > 0$, $x = 0$ or $x < 0$. If $x = 0$, then $x^2 = 0$ so we are done. If $x > 0$, then by O4, $x^2 > 0 \cdot x = 0$ and we are done. If $x < 0$, then, by O3, $x - x < 0 - x$ and so $-x > 0$. So $x^2 = (-x)^2 > 0$ by the second case.

(b) (5 points) Using part (a), or otherwise, prove that $x^2 + \frac{1}{x^2} \geq 2$ for all $x \in \mathbb{R}$ with $x \neq 0$.

Solution: Note that $x^2 - 2 + \frac{1}{x^2} = (x - \frac{1}{x})^2 \geq 0$ by part (a). Therefore (adding 2 to both sides using O3) $x^2 + \frac{1}{x^2} \geq 2$ as required.

3. (a) (6 points) Prove by induction on n that, for $n \geq 1$,

$$\sum_{i=1}^{n} i2^i = 2 + (n-1)2^{n+1}. \tag{1}$$

The notation $\sum_{i=1}^{n} i2^i$ *means* $2^1 + 2 \cdot 2^2 + \ldots + n2^n$.

Solution: Base case. Suppose that $n = 1$. Then

$$\sum_{i=1}^{n} i2^i = 1 \cdot 2^1 = 2$$

and $2 + (n-1)2^{n+1} = 2 + 0 \cdot 2^1 = 2$. These are equal as required.

Induction step Suppose that equation (1) is true for n. We must show that it is true for $n+1$. But

$$\begin{aligned} \sum_{i=1}^{n+1} i2^i &= \sum_{i=1}^{n} i2^i + (n+1)2^{n+1} \\ &= 2 + (n-1)2^{n+1} + (n+1)2^{n+1} && \text{by the induction hypothesis} \\ &= 2 + 2n2^{n+1} \\ &= 2 + n2^{n+2} && \text{as } 2 \cdot 2^{n+1} = 2^{n+2} \\ &= 2 + (n+1-1)2^{(n+1)+1}. \end{aligned}$$

So equation (1) is true for $n+1$ as required.

(b) (4 points) Prove that the function $f(x) = \frac{x}{x+1}$ is injective.

Solution: Suppose that $f(x) = f(y)$. We must show that $x = y$. But

$$\begin{aligned} \frac{x}{x+1} &= \frac{y}{y+1} \\ \therefore x(y+1) &= y(x+1) && \text{multiplying up} \\ \therefore xy + x &= yx + y \\ \therefore x &= y && \text{cancelling } xy, \end{aligned}$$

as required.

4. (a) (5 points) Prove from the definition that

$$\lim_{x \to a} x^2 = a^2.$$

Solution: Note first that, if $|x - a| < \delta \leq 1$, then

$$\begin{aligned} |x^2 - a^2| &= |x - a||x + a| \\ &\leq \delta|x + a| \\ &\leq \delta(|x| + |a|) && \text{triangle ineq.} \\ &< \delta(|a| + 1 + |a|) \\ &= \delta(2|a| + 1). \end{aligned}$$

In the penultimate step we have used that $|x| = |x-a+a| \leq |x-a|+|a| < 1+|a|$. So let $\epsilon > 0$ and take $\delta = \min(\frac{\epsilon}{2|a|+1}, 1)$. Then we have shown that, if $|x-a| < \delta$, then

$$|x^2 - a^2| < \delta(2|a| + 1) \leq \epsilon.$$

So $\lim_{x \to a} x^2 = a^2$ as required.

(b) (5 points) Calculate

$$\lim_{x \to 1} \frac{1 - x^3}{1 - x^2}.$$

You may use any results from class.

Solution: Note that $1 - x^3 = (1 - x)(1 + x + x^2)$ and $1 - x^2 = (1 - x)(1 + x)$. So

$$\frac{1 - x^3}{1 - x^2} = \frac{1 + x + x^2}{1 + x}.$$

Now, $\lim_{x\to 1} 1 = 1$, $\lim_{x\to 1} x = 1$, and $\lim_{x\to 1} x^2 = 1^2 = 1$. So by algebra of limits,

$$\lim_{x\to 1}(1 + x + x^2) = \lim_{x\to 1} 1 + \lim_{x\to 1} x + \lim_{x\to 1} x^2 = 1 + 1 + 1 = 3$$

and similarly $\lim_{x\to 1}(1 + x) = 2$. As $\lim_{x\to 1}(1 + x) \neq 0$, we can apply the quotient part of algebra of limits to get

$$\lim_{x\to 1} \frac{1 + x + x^2}{1 + x} = \frac{\lim_{x\to 1}(1 + x + x^2)}{\lim_{x\to 1}(1 + x)} = \frac{3}{2}.$$

So the limit is $\frac{3}{2}$.