Midterm solutions

1a) If $P = \{t_0, t_1, \ldots, t_n\}$ with $t_0 < t_1 < \ldots < t_n$,

then $U(f, P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sup f([t_{i-1}, t_i])$.

$\underline{U}(f) = \inf \{ U(f, P) : P$ a partition of $[a, b] \}$.

b) Let P_n be the partition $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\}$.

Then (if $f(x) = x^2$)

$$L(f, P_n) = \sum_{i=1}^{n} \frac{1}{n} \times \left(\frac{i-1}{n}\right)^2$$

$$= \frac{1}{n^3} \sum_{i=0}^{n-1} i^2$$

$$= \frac{n(n-1)(2n-1)}{6n^3}$$

$$= \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}.$$

$\therefore \underline{L}(f) \geq \sup \{ \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} : n \in \mathbb{N} \}$

$= \frac{1}{3}$.

Similarly, $U(f, P_n) = \sum_{i=1}^{n} \frac{1}{n} \times \left(\frac{i}{n}\right)^2$

$$= \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$\therefore \underline{U}(f) \leq \frac{1}{3}$.

But, always, $\underline{L}(f) \leq \underline{U}(f)$. So $\frac{1}{3} \leq \underline{L}(f) \leq \underline{U}(f) \leq \frac{1}{3}$, so $\underline{L}(f) = \underline{U}(f) = \frac{1}{3}$.

so f is integrable &

$$\boxed{\int_0^1 x^2\,dx = \frac{1}{3}}$$

2a) NO ~~(example from class)~~

b) YES &

c) YES

d) YES

e) NO.

3a) We have to show that

$$\lim_{x\to c} F(x) = F(c)$$

i.e. $$\lim_{x\to c} \left(F(x) - F(c)\right) = 0$$

i.e. $$\lim_{x\to c} \int_a^x f - \int_a^c f = 0$$

i.e. $$\lim_{x\to c} \int_c^x f = 0, \quad \text{as } \int_a^x f = \int_a^c f + \int_c^x f.$$

As f is integrable, it is bounded, so $|f| \le M$ for some M.

~~Then As $|f| \le M$ for all~~

$$\therefore \left|\int_c^x f\right| \le |x-c| \cdot M$$

So $0 \le \left|\int_c^x f\right| \le M \cdot |x-c|$ for all x; as

M-test $\lim_{x \to c} M \cdot |x-c| = 0$, get

$$\lim_{x \to c} \left|\int_c^x f\right| = 0 \quad \text{(squeeze theorem)}$$

$$\therefore \lim_{x \to c} \int_c^x f = 0 \quad \text{as required.}$$

b) $G(x) = F(f(x))$, $F(x) = \int_0^x f$. $F'(x) = f$ by ~~the~~ FTC1 as f is cts

$\therefore \; G'(x) = f'(x) F'(f(x))$ chain rule, as f differentiable

$$= f'(x) f(f(x)).$$

4a) Def^n if $x > 0$, $\log(x) = \int_1^x \frac{1}{t}\,dt$.

Then $\log(1) = \int_1^1 \frac{1}{t}\,dt = 0$

& $\log'(x) = \frac{1}{x}$ by FTC 1.

Claim $\forall x, y > 0$, $\boxed{\log(xy) = \log(x) + \log(y)}$

Proof Fix $y > 0$. When $x = 1$,

$$\begin{aligned} \log(xy) &= \log(y) \\ &= \log(1) + \log(y) \\ &= \log(x) + \log(y). \end{aligned}$$

So, to prove that $\log(xy) = \log(x) + \log(y)$, STP that

$$\cancel{\tfrac{d}{dx}}\,(\log(xy))' = (\log(x) + \log(y))'$$

(differentiating wrt x.)

But $(\log(xy))' = y \times \dfrac{1}{xy}$ chain rule

$$= \frac{1}{x}$$

$$= \log'(x)$$

$$= (\log(x) + \log(y))' \quad \text{as } y \text{ is constant.} \quad \checkmark .$$

b) By FTC1, $C'(x) = \cos(x^2)$.

$\therefore \ (xC(x))' = x\cos(x^2) + C(x)$.

Note that $x\cos(x^2) = \left(\frac{1}{2}\sin(x^2)\right)'$

$\therefore \ x\,C(x) - \frac{1}{2}\sin(x^2)$ is an antiderivative of $C(x)$ $\square$.