BASIC PROPERTIES OF INTEGRALS.

Let $A \subset \mathbb{R}^n$ be a closed rectangle and let $f, g : A \to \mathbb{R}$ be bounded functions.

Theorem 0.1. (1) *If $c \in \mathbb{R}$ then $\int_A c = c \operatorname{vol} A$.*

(2) *If f, g are integrable, so is $f + g$ and*

$$\int_A f + g = \int_A f + \int_A g.$$

(3) *If f is integrable and $c \in \mathbb{R}$, so is cf, and*

$$\int_A cf = c \int_A f.$$

(4) *If $f \geq g$ are integrable, then*

$$\int_A f \geq \int_A g.$$

(5) *If f is integrable, so is $|f|$, and*

$$\int_A |f| \geq |\int_A f|.$$

(6) *If f is integrable and $|f| \leq M$, then $|\int_A f| \leq \operatorname{vol}(A) \cdot M$.*

(7) *If f is integrable, so are $f_+ = \max(f, 0)$ and $f_- = \min(f, 0)$. Conversely, if f_+ and f_- are integrable, so is f.*

(8) *If f and g are integrable, so is fg.*

Proof. (1) If $\mathcal{P}$ is any partition,

$$L(f, \mathcal{P}) = U(f, \mathcal{P}) = \sum_{S \vdash \mathcal{P}} \operatorname{vol}(S)c = c \operatorname{vol}(A).$$

(2) For any $\mathcal{P}$, since $U(f + g, \mathcal{P}) \leq U(f, \mathcal{P}) + U(g, \mathcal{P})$ for all $\mathcal{P}$, $\underline{U}_A(f + g) \leq \underline{U}_A(f) + \underline{U}_A(g) = \int_A f + \int_A g$. Similarly for L. So

$$\int_A f + \int_A g \leq \underline{L}_A(f + g) \leq \underline{U}_A(f + g) \leq \int_A f + \int_A g$$

and we have equality throughout, giving the desired result.

(3) If $c \geq 0$, then $L(cf, \mathcal{P}) = cL(f, \mathcal{P})$ and similarly for U and the result follows. If $c < 0$, then $L(cf, \mathcal{P}) = cU(f, \mathcal{P})$ (since inequalities are reversed by multiplying by c) and similarly for U and the result still follows.

(4) Since $U(f, \mathcal{P}) \geq U(g, \mathcal{P})$ for any $\mathcal{P}$,

$$\int_A f = \underline{U}_A(f) \geq \underline{U}_A(g) = \int_A g.$$

(5) The key inequality is $\sup |f|(S) - \inf |f|(S) \leq \sup f(S) - \inf f(S)$, exercise. From this it follows that

$$(U - L)(|f|, \mathcal{P}) \leq (U - L)(f, \mathcal{P})$$

and so if f is integrable, then the right hand side, and so the left hand side, can be made arbitrarily small and $|f|$ is integrable.

Since $-|f| \leq f \leq |f|$, parts 3 and 4 imply that

$$-\int_A |f| \leq \int_A f \leq \int_A |f|$$

and so

$$|\int_A f| \leq \int_A |f|.$$

(6) Let $|f| \leq M$ for all M. Applying parts 5, 4 and 1,

$$|\int_A f| \leq \int_A |f| \leq \int_A M = M \operatorname{vol} A.$$

(7) As $f_+ = \frac{f+|f|}{2}$ and $f_- = \frac{f-|f|}{2}$ and $f = f_+ + f_-$, this all follows from parts 2 and 4.

(8) Firstly, suppose that $f, g \geq 0$. Choose M an upper bound for f and g. If $x, y \in A$ then (exercise)

$$|(fg)(x) - (fg)(y)| \leq M\left(|f(x) - g(x)| + |f(y) - g(y)|\right).$$

So if $S \vdash \mathcal{P}$ then

$$\sup(fg)(S) - \inf(fg)(S) \leq M\left((\sup f(S) - \inf f(S)) + (\sup g(S) - \inf g(S))\right).$$

This implies that if $\epsilon > 0$ and we choose $\mathcal{P}$ so that $(U-L)(f,\mathcal{P}) < \epsilon$ and $(U-L)(g,\mathcal{P}) < \epsilon$ then

$$(U-L)(fg,\mathcal{P}) < 2M\epsilon$$

which can be made arbitrarily small, so fg is integrable.

Now, part 3 with $c = -1$ implies that the same result holds if $f \leq 0$ and $g \geq 0$, or $f \geq 0$ and $g \leq 0$, or $f \leq 0$ and $g \leq 0$.

In general,

$$fg = (f_+ + f_-)(g_+ + g_-) = f_+g_+ + f_+g_- + f_-g_+ + f_-g_-$$

which is integrable by parts 6 and 2, together with the cases we just proved (as $f_+, g_+ \geq 0$, $f_-, g_- \leq 0$ by definition).

□