MATH 159 – HOMEWORK 9

This homework is not graded. All vector spaces are finite dimensional over F, unless otherwise stated.

(1) (a) If $T : V \to W$ is a linear map, $\mathcal{B}$ and $\mathcal{C}$ are ordered bases of V and W respectively, and $v \in V$, show that

$$[Tv]_{\mathcal{C}} = [T]_{\mathcal{C},\mathcal{B}}[v]_{\mathcal{B}}.$$

(b) If $\mathcal{B}'$ is another ordered basis for v, derive a formula for $[v]_{\mathcal{B}'}$ in terms of $[v]_{\mathcal{B}}$ and the change of basis matrix.

(2) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = 2x$. Write down the matrix of T with respect to the ordered basis $((1, 2), (-2, 1))$. Use the change of basis formula to find the matrix of T with respect to the standard basis of $\mathbb{R}^2$.

(3) (a) Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$.

(b) Show that it's impossible to find a 2×3 matrix A and a 3×2 matrix B such that $BA = I_3$.

(4) Define the **trace** of an $n \times n$ matrix to be the sum of its diagonal entries; that is, for $A = (a_{ij})$,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $n \times n$ matrices A and B. Deduce that, if A and B are similar, they have the same trace.

(5) Find the determinant of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(6) Prove that, if A is an upper-triangular $n \times n$ matrix, then its determinant is the product of its diagonal entries.

(7) (optional, hard!) For $n \geq 1$, let D_n the largest possible absolute value of $|\det A|$ for A a real $n \times n$ matrix all of whose entries are ± 1. Work out D_n for $n = 1, 2, 3, 4$. Can you say anything for general n?