MATH 163 – HOMEWORK 8

Due at noon on May 31st.

(1) Let $A = [0,1] \times [0,1]$. Direct from the definition, find $\int_A f$ for the following functions f:
  (a) $f(x,y) = 1$ if $x = y$, 0 otherwise;
  (b) $f(x,y) = |x - y|$;
  (c) $f(x,y) = 1$ if $x + y \leq 1$, 0 otherwise.

(2) Suppose that $g, h : [0,1] \to \mathbb{R}$ are non-negative and increasing (and therefore integrable, by Spivak Calculus 13-20). Prove from the definition that $f(x,y) = g(x)h(y)$ is integrable on $[0,1]^2$ and

$$\int_{[0,1]^2} f = \left(\int_0^1 g\right)\left(\int_0^1 h\right).$$

Prove the same thing again using Fubini's theorem.

(3) Show that the rational numbers in $[0,1]$ do not have content 0. Write down a function f on $[0,1]$ that is integrable, but such that the set of discontinuities of f does not have content zero.

(4) Suppose that X is a set of content zero contained in a closed rectangle A. Prove that 1_X is integrable and $\int_A 1_X = 0$. Deduce that, if f and g are integrable functions on A that are equal except on a set of content zero, then $\int_A f = \int_A g$.

(5) Let $f : A \to \mathbb{R}$ be a non-negative, continuous function on a closed rectangle $A \subset \mathbb{R}^n$. Let

$$C = \{(a,y) \in \mathbb{R}^{n+1} : a \in A, 0 \leq y \leq f(a)\}.$$

  (a) Show that 1_C is integrable (on a large enough closed rectangle $B \subset \mathbb{R}^{n+1}$).
  (b) Use Fubini to show that $\int_B 1_C = \int_A f$.

(6) Spivak 3-27.

(7) Compute the following integrals:
  (a) $\int_{[-1,1]\times[0,2]} x + y$;
  (b) $\int_{[0,1]\times[0,1]\times[0,1]} xyz$;
  (c) $\int_{[1,2]\times[0,1]} \frac{1}{x+y}$;
  (d) $\int_{[0,\pi/2]\times[0,\pi/2]} \sin(x+y)$;
  (e) $\int_T (1 - x - y)$ where $T \subset \mathbb{R}^2$ is the triangle $0 \leq x \leq 1$, $0 \leq y \leq 1$, $x + y \leq 1$;
  (f) $\int_1^2 \int_1^x \frac{x^2}{y^2} dy dx$.

(8) Suppose that the cube $[0,1] \times [0,1] \times [0,1]$ is made of a substance whose density at a point (x,y,z) is $f(x,y,z) = x\sqrt{y+z}$. Find the mass of the cube.

(9) Let $S_n(r) = \{x \in \mathbb{R}^n : |x| \leq r\}$, the closed n-dimensional sphere of radius r. If the volume of $S_n(r)$ is $C_n r^n$ for some constant C_n, show the following:
  (a) $C_1 = 2$;

(b) $C_2 = \pi$;
(c) $C_{n+1} = \int_{-1}^{1} C_n(\sqrt{1-x^2})^n dx$;
(d) Deduce that $C_{n+1} = C_n \int_0^{\pi} \sin^{n+1}(\theta) d\theta$.
(e) Use the formulae of Spivak problem 19-41(b) to show that

$$C_{n+2} = \frac{2\pi}{n+2} C_n.$$

(f) Find closed formulae for C_n when n is odd and when n is even. Check your answer when $n = 3$. Write down the volume of a 4-dimensional sphere.