MATH 159 – HOMEWORK 8

Due 2pm on March 2nd. Submit all problems; only the starred problems are graded. Throughout, F denotes a field.

(1) (*)Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let

$$T((x,y) = (x-y, x+y, 2x-3y)$$

for $(x,y) \in \mathbb{R}^2$. Find the matrix of T with respect to each of the following pairs of bases:

(a) $\mathcal{B} = ((1,0),(0,1))$, $\mathcal{C} = ((1,0,0),(0,1,0),(0,0,1))$;

(b) $\mathcal{B} = ((1,1),(-1,1))$, $\mathcal{C} = ((1,1,1),(1,1,0),(1,0,0))$.

(2) (*)Let $V = \mathbb{R}^2$, and let T be the linear map given by reflection in the line $y = 2x$. Find the matrix of T with respect to the following bases:

(a) $\mathcal{B} = ((1,0),(0,1))$;

(b) $\mathcal{B} = ((1,2),(-2,1))$.

Think about why the matrix is nicer in the second basis.

(3) (*)Let $U = \mathbb{R}^2$, $V = \mathbb{R}^3$, and $W = \mathbb{R}^2$. Let $\mathcal{A} = ((1,0),(0,1))$, $\mathcal{B} = ((1,0,0),(0,1,0),(0,0,1))$ and $\mathcal{C} = ((1,1),(1,-1))$. Let $S : U \to V$ be the map

$$(x,y) \mapsto (x+y, x-y, 2y)$$

and let $T : V \to W$ be the map

$$(x,y,z) \mapsto (x+y+z, x-y-z).$$

(a) Find the matrices $[S]_{\mathcal{B},\mathcal{A}}$ and $[T]_{\mathcal{C},\mathcal{B}}$.

(b) Show that $(T \circ S)((x,y)) = (2x+2y, 0)$ and hence compute the matrix $(T \circ S)_{\mathcal{C},\mathcal{A}}$.

(c) Verify that $[T]_{\mathcal{C},\mathcal{B}}[S]_{\mathcal{B},\mathcal{A}} = [T \circ S]_{\mathcal{C},\mathcal{A}}$.

(4) Let $V = \mathbb{R}^2$. Let $0^\circ \leq \theta, \phi < 360^\circ$. Let

$$R_\theta, R_\phi : V \to V$$

be rotation by θ and by ϕ anticlockwise. Write down their matrices with respect to the standard basis of $\mathbb{R}^2$, and using the fact that

$$R_\theta \circ R_\phi = R_{\theta+\phi}$$

derive the addition formulae for sin and cos.

(5) An $n \times n$ matrix $A = (a_{ij})$ is **upper triangular** if all the entries below the diagonal are zero, i.e. if $a_{ij} = 0$ for $i > j$. Show that, if A and B are upper-triangular $n \times n$ matrices, then so is AB.

(6) (*)Let $V = \mathbb{R}[X]_3$, the vector space of polynomials of degree 3, and let $\mathcal{B}$ be the ordered basis $(1, X, X^2, X^3)$. Let $D : V \to V$ be the linear map given by differentiation (see hw7 q9). Find the matrix A of D with respect to $\mathcal{B}$. Find A^2, A^3 and A^4.