

MATH 159 – HOMEWORK 8

Due 1pm on November 25th. Submit all problems; only the starred problems are graded. Throughout, F denotes a field.

- (1) (*) Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let

$$T((x, y)) = (x - y, x + y, 2x - 3y)$$

for $(x, y) \in \mathbb{R}^2$. Find the matrix of T with respect to each of the following pairs of bases:

- (a) $\mathcal{B} = ((1, 0), (0, 1))$, $\mathcal{C} = ((1, 0, 0), (0, 1, 0), (0, 0, 1))$;
 (b) $\mathcal{B} = ((1, 1), (-1, 1))$, $\mathcal{C} = ((1, 1, 1), (1, 1, 0), (1, 0, 0))$.
 (2) (*) Let $V = \mathbb{R}^2$, and let T be the linear map given by reflection in the line $x = y$. Find the matrix of T with respect to the following bases:
 (a) $\mathcal{B} = ((1, 0), (0, 1))$;
 (b) $\mathcal{B} = ((1, 1), (1, -1))$.

Think about why the matrix is nicer in the second basis.

- (3) (*) Let $U = \mathbb{R}^2$, $V = \mathbb{R}^3$, and $W = \mathbb{R}^2$. Let $\mathcal{A} = ((1, 0), (0, 1))$, $\mathcal{B} = ((1, 0, 0), (0, 1, 0), (0, 0, 1))$ and $\mathcal{C} = ((1, 1), (1, -1))$. Let $S : U \rightarrow V$ be the map

$$(x, y) \mapsto (x + y, x - y, 2y)$$

and let $T : V \rightarrow W$ be the map

$$(x, y, z) \mapsto (x + y + z, x - y - z).$$

- (a) Find the matrices $[S]_{\mathcal{B}, \mathcal{A}}$ and $[T]_{\mathcal{C}, \mathcal{B}}$.
 (b) Show that $(T \circ S)((x, y)) = (2x + 2y, 0)$ and hence compute the matrix $(T \circ S)_{\mathcal{C}, \mathcal{A}}$.
 (c) Verify that $[T]_{\mathcal{C}, \mathcal{B}}[S]_{\mathcal{B}, \mathcal{A}} = [T \circ S]_{\mathcal{C}, \mathcal{A}}$.
 (4) Let $V = \mathbb{R}^2$. Let $0^\circ \leq \theta, \phi < 360^\circ$. Let

$$R_\theta, R_\phi : V \rightarrow V$$

be rotation by θ and by ϕ anticlockwise. Write down their matrices with respect to the standard basis of \mathbb{R}^2 , and using the fact that

$$R_\theta \circ R_\phi = R_{\theta + \phi}$$

derive the addition formulae for sin and cos.

- (5) An $n \times n$ matrix $A = (a_{ij})$ is **upper triangular** if all the entries below the diagonal are zero, i.e. if $a_{ij} = 0$ for $i > j$. Show that, if A and B are upper-triangular $n \times n$ matrices, then so is AB .
 (6) (*) Let $V = \mathbb{R}[X]_3$, the vector space of polynomials of degree 3, and let \mathcal{B} be the ordered basis $(1, X, X^2, X^3)$. Let $D : V \rightarrow V$ be the linear map given by differentiation (see hw7 q9). Find the matrix A of D with respect to \mathcal{B} . Find A^2 , A^3 and A^4 .
 (7) Let V and W be finite-dimensional vector spaces, of dimensions n and m respectively. Let $L(V, W)$ be the set of linear maps $V \rightarrow W$. Show that $L(V, W)$ is naturally a vector space (you should define the addition and scalar multiplication). Find its dimension.