MATH 159 – HOMEWORK 8

This week's homework is not graded.

(1) Prove that the function $f : [0, 2] \to \mathbb{R}$ given by $f(x) = 0$ for $0 \leq x \leq 1$ and $f(x) = 1$ for $1 < x \leq 2$ is integrable, and find its integral.

(2) Prove that, if $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ integrable, then $f + g$ is integrable with integral

$$\int_A f + g = \int_A f + \int_A g.$$

(3) Prove that, if $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are integrable, then fg is integrable. *Hint: First, note that $|f(x)|$ and $|g(x)|$ are bounded by some M and show that, if S is a subrectangle of a partition of A and $x, y \in S$, then*

$$f(x)g(x) - f(y)g(y) \leq M(|f(x) - f(y)| + |g(x) - g(y)|).$$

(4) Let $A = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, let $T = \{(x, y) \in A : x + y \leq 1\}$, and let $f : A \to \mathbb{R}$ be the indicator function of T — that is, let

$$f(x, y) = \begin{cases} 1 \text{ if } (x, y) \in T \\ 0 \text{ if } (x, y) \notin T \end{cases}.$$

Show that, for every $\epsilon > 0$, there is a partition P of A such that the total area of the subrectangles of P that intersect the line $x + y = 1$ is $< \epsilon$. Deduce that f is integrable over A, and that $\int_A f = \frac{1}{2}$.

(5) Let $A \subset \mathbb{R}^n$ be a rectangle, and suppose that $f, g : A \to \mathbb{R}$ are functions such that $f(x) = g(x)$ for all except a finite number of x. Show that, if f is integrable, then so is g, and

$$\int_A f = \int_A g.$$

(6) Compute the following integrals:
   (a) $\int_{[-1,1]\times[0,2]} x + y$;
   (b) $\int_{[0,1]\times[0,1]\times[0,1]} xyz$;
   (c) $\int_{[1,2]\times[0,1]} \frac{1}{x+y}$;
   (d) $\int_{[0,\pi/2]\times[0,\pi/2]} \sin(x + y)$;
   (e) $\int_T (1 - x - y)$ where $T \subset \mathbb{R}^2$ is the triangle $0 \leq x \leq 1$, $0 \leq y \leq 1$, $x + y \leq 1$;
   (f) $\int_1^2 \int_1^x \frac{x^2}{y^2} dydx$.

(7) Let $S \subset \mathbb{R}^3$ be defined by $x^2 + y^2 + z^2 \leq 1$, and A some rectangle containing S. Assuming that $\int_S 1$ is defined, show that it can be calculated as:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1 - x^2 - y^2}\, dydx.$$

By evaluating this integral, find the volume of S.

(8) Suppose that the cube $C = [0,1] \times [0,1] \times [0,1]$ is made of a substance whose density at a point (x, y, z) is $f(x, y, z) = x\sqrt{y+z}$. Find the mass of the cube; that is, find

$$\int_C f.$$