MATH 163 – HOMEWORK 7

Due noon on May 26th.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}^2$, and $h : \mathbb{R}^2 \to \mathbb{R}$ be the linear maps defined by

$$\begin{aligned} f(x,y) &= (x+y, x-y, 2x+3y) \\ g(x,y,z) &= (x-y-z, y+2z) \\ h(x,y) &= x-y. \end{aligned}$$

Find the matrices of the following compositions (note: one of them does not exist!):
(a) $f \circ g$;
(b) $g \circ f$;
(c) $f \circ h$;
(d) $h \circ g \circ f$.

(2) Spivak 2-17 b,d,f,h.

(3) Spivak 2-19.

(4) Spivak 2-28 a,c.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}.$$

In homework 6 you proved that f is not differentiable at $(0,0)$. Prove that the partial derivatives of f do exist at $(0,0)$. Why does this not contradict the theorem from class?

(6) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ has a minimum or a maximum at a and is differentiable at a. Prove that $Df(a) = 0$. (Use partial derivatives and the corresponding fact for functions $\mathbb{R} \to \mathbb{R}$).

(7) Show that $f(x,y) = x^4+y^4+4xy$ has a minimum in $\mathbb{R}^2$ and use the previous question to find the minimum value.

(8) If $f(x,y) = x^2 - y^2$, show that $Df(0,0)$ is zero but that $(0,0)$ is neither a maximum or a minimum.

(9) If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at x with derivative $Df(x)$ and if $u \in \mathbb{R}^n$ define the **directional derivative** $D_u f(x)$ to be

$$\lim_{h \to 0} \frac{f(x+hu) - f(x)}{h}.$$

It is the rate of change of f as you move in the u direction at speed $|u|$.
Prove that $D_u f(x)$ exists, and is equal to $Df(x)(u)$.

(10) In the setup of the previous question, suppose that

$$f'(x) = (a_1 \quad a_2 \quad \dots \quad a_n)$$

is non-zero. Let $u = (a_1, \ldots, a_n)$ and let $\hat{u}$ be the unit vector in the u direction. Prove that, for any other unit vector $\hat{v}$,

$$D_{\hat{v}} f(x) \leq D_{\hat{u}} f(x).$$

Use this to give a geometric interpretation of $f'(x)$.

(11) If $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, then let

$$S_f = \{x \in \mathbb{R}^3 : f(x) = 0\}.$$

If $a \in S_f$ and $Df(a)$ is non-zero, then the **tangent space** of S_f at a is defined to be

$$T_a S_f = \{v \in \mathbb{R}^3 : Df(a)(v) = 0\}.$$

It is the set of directions which are tangent to the surface S_f at a. (note that the tangent space won't usually pass through a; it is translated to pass through the origin)

Prove that, if $v, w \in T_a S_f$, then so are $v + w$ and λv (for $\lambda \in \mathbb{R}$).

(12) For each of the following functions f, draw S_f (use a computer), and find the tangent space at the given point a.
(a) $f(x, y, z) = x^2 + y^2 + z^2 - 14$, $a = (3, 2, 1)$;
(b) $f(x, y, z) = x^2 + y^2 - z^2 - 1$, $a = (1, 1, 1)$;
(c) $f(x, y, z) = x^3 + y^3 + z^3 - 1 - 4xyz$, $a = (2, 2, 1)$.
(d) $f(x, y, z) = x^2 + y^2 - xyz - 4$, $a = (3, 1, 2)$.