MATH 159 – HOMEWORK 7

Due 2pm on February 25th. Submit all problems; only the starred problems are graded. Throughout, F denotes a field.

(1) (*) Find a basis of $\mathbb{R}^4$ containing the vectors $(0, 1, 2, 3)$ and $(0, 1, 0, 1)$.

(2) Prove that, if V is an n-dimensional vector space and $S \subset V$ is a linearly independent subset of size m, then $\dim V \geq m$.

(3) (*) Suppose that W_1 and W_2 are subspaces of a finite-dimensional vector space V, and that $\dim W_1 + \dim W_2 > \dim V$. Show that $W_1 \cap W_2 \neq \{\underline{0}\}$.

(4) Let $F[X]_n$ be the vector space of polynomials of degree at most n over F; that is,

$$F[X]_n = \{a_0 + a_1 X + \ldots + a_n X^n \mid a_0, \ldots, a_n \in F\}$$

with addition and scalar multiplication being given by that for polynomials. Show that it has dimension $n + 1$.

(5) (*) Show that, if $T : V \to W$ is a bijective linear map, then its inverse T^{-1} is also a linear map.

(6) Show that, if $T : V \to W$ is an injective linear map, then $\dim W \geq \dim V$.

(7) (*) Show that a linear map $T : V \to W$ is injective if and only if its kernel is $\{\underline{0}\}$.

(8) (a) Let T be the linear map $\mathbb{R}^3 \to \mathbb{R}^2$ given by

$$(x, y, z) \mapsto (x + y + z, x - y).$$

Give bases for its image and its kernel.

(b) (*)Let T be the linear map $\mathbb{R}^4 \to \mathbb{R}^3$ given by

$$(w, x, y, z) \mapsto (x + 2y + z, z - w, 2x + 4y + 2w).$$

Give bases for its image and kernel.

(9) (*) Let $D : \mathbb{R}[X]_n \to \mathbb{R}[X]_{n-1}$ (cf q4) be given by "differentiation", that is:

$$D(a_0 + a_1 X + a_2 X^2 + \ldots a_n X^n) = a_1 + 2a_2 X + \ldots + n a_n X^{n-1}.$$

Show that D is linear, and find its kernel and image. (optional) What happens if the field $\mathbb{R}$ is replaced by the field $\mathbb{Z}_p$ for a prime p?

(10) You will need to use the rank-nullity theorem for this question.

(a) Find a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T^2 = 0$ but $T \neq 0$. Show that the image and kernel of any such T must be one-dimensional.

(b) Find a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $T^4 = 0$ but $T^3 \neq 0$. Show that the kernel of any such T must be one-dimensional.

(c) Find a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $T^2 = T$ and the image and kernel of T are both two-dimensional. Show that the image and kernel of any such T must have trivial intersection (i.e. $\mathrm{im} T \cap \ker T = \{\underline{0}\}$).