

MATH 159 – HOMEWORK 7

Due 1pm on November 18th. Submit all problems; only the starred problems are graded. Throughout, F denotes a field.

- (1) (*) Find a basis of \mathbb{R}^4 containing the vectors $(0, 1, 2, 3)$ and $(0, 1, 0, 1)$.
- (2) Prove that, if V is an n -dimensional vector space and $S \subset V$ is a linearly independent subset of size m , then $\dim V \geq m$.
- (3) (*) Suppose that W_1 and W_2 are subspaces of a finite-dimensional vector space V , and that $\dim W_1 + \dim W_2 > \dim V$. Show that $W_1 \cap W_2 \neq \{0\}$.
- (4) Let $F[X]_n$ be the vector space of polynomials of degree at most n over F ; that is,

$$F[X]_n = \{a_0 + a_1X + \dots + a_nX^n \mid a_0, \dots, a_n \in F\}$$

with addition and scalar multiplication being given by that for polynomials. Show that it has dimension $n + 1$.

- (5) Show that, if $T : V \rightarrow W$ is a bijective linear map, then its inverse T^{-1} is also a linear map.
- (6) Show that, if $T : V \rightarrow W$ is an injective linear map, then $\dim W \geq \dim V$.
- (7) (*) Show that a linear map $T : V \rightarrow W$ is injective if and only if its kernel is $\{0\}$.
- (8) (a) Let T be the linear map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$(x, y, z) \mapsto (x + y + z, x - y).$$

Give bases for its image and its kernel.

- (b) (*) Let T be the linear map $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by

$$(w, x, y, z) \mapsto (x + 2y + z, z - w, 2x + 4y + 2w).$$

Give bases for its image and kernel.

- (9) (*) Let $D : \mathbb{R}[X]_n \rightarrow \mathbb{R}[X]_{n-1}$ (cf q4) be given by “differentiation”, that is:

$$D(a_0 + a_1X + a_2X^2 + \dots + a_nX^n) = a_1 + 2a_2X + \dots + na_nX^{n-1}.$$

Show that D is linear, and find its kernel and image. (optional) What happens if the field \mathbb{R} is replaced by the field \mathbb{Z}_p for a prime p ?

- (10) (a) Find a linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T^2 = 0$ but $T \neq 0$. Show that the image and kernel of any such T must be one-dimensional.
- (b) (*) Find a linear map $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that $T^4 = 0$ but $T^3 \neq 0$. Show that the kernel of any such T must be one-dimensional.
- (c) Find a linear map $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that $T^2 = T$ and the image and kernel of T are both two-dimensional. Show that the image and kernel of any such T must have trivial intersection (i.e. $\text{im}T \cap \ker T = \{0\}$).