MATH 159 – HOMEWORK 6

Due 2pm on May 11th. Submit all problems; only the starred problems are graded. You can assume standard derivatives of functions $\mathbb{R} \to \mathbb{R}$.

(1) (*) For each of the following functions $\mathbb{R}^n \to \mathbb{R}^m$, say whether or not it is linear (no proof needed), and if it is linear write down its matrix.
   (a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$;
   (b) $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto 3x - 2y$;
   (c) $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x^2 + y^2 + z^2$;
   (d) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x + 1, y + 1)$;
   (e) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x + y, 2x - 3y - z)$;
   (f) $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (3x, 7x)$;
   (g) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 3x + 5$;
   (h) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(x, y) \mapsto (3x - 5y, x + y, 4y)$.

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}^2$, and $h : \mathbb{R}^2 \to \mathbb{R}$ be the linear maps defined by

$$\begin{aligned} f(x, y) &= (x + y, x - y, 2x + 3y) \\ g(x, y, z) &= (x - y - z, y + 2z) \\ h(x, y) &= x - y. \end{aligned}$$

Find the matrices of the following compositions (note: one of them does not exist!):
   (a) $f \circ g$;
   (b) $g \circ f$;
   (c) $f \circ h$;
   (d) $h \circ g \circ f$.

(3) (*) Let $U = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$. Prove from the definition that $f : U \to \mathbb{R}$ defined by $f(x, y) = \frac{x}{y}$ is differentiable at $(1, 1)$ with derivative

$$(h, k) \mapsto h - k.$$

(4) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is of the form $f(x, y) = g(x)$ for some function g that is differentiable everywhere. Prove that f is differentiable everywhere and that, at a point $(x, y) \in \mathbb{R}^2$,

$$Df(x, y)(h, k) = Dg(x)(h).$$

(5) Show that the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = |x|$ is not differentiable at 0.

(6) (*) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} \text{ if } (x, y) \neq (0, 0) \\ 0 \text{ if } (x, y) = (0, 0) \end{cases}$$

is continuous at 0 but is not differentiable there (hint: consider $f(h, h)$ as $h \to 0$ and $f(0, h)$ as $h \to 0$).

(7) (*) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(x)| \leq |x|^2$ for all $x \in \mathbb{R}^n$. Prove that f is differentiable at $(0, \ldots, 0)$.

(8) Use the chain rule to prove that, if f and g are functions $\mathbb{R}^n \to \mathbb{R}$, then

$$D(f+g)(x) = Df(x) + Dg(x).$$

(9) Use the chain rule to prove that, if f and g are functions $\mathbb{R}^n \to \mathbb{R}$, then

$$D(fg)(x) = g(x)Df(x) + f(x)Dg(x).$$

(10) (*) Using theorems from class and known one-variable derivatives, calculate the derivatives of the following.

(a) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xyz$;
(b) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xy + yz + zx$;
(c) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x + y + z$;
(d) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x+y+z, xy+yz+zx, xyz)$;
(e) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x\cos y, x\sin y)$;
(f) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x\cos y + x\sin y$.