MATH 163 – HOMEWORK 6

Due noon on May 15th.

(1) For each of the following functions $\mathbb{R}^n \to \mathbb{R}^m$, say whether or not it is linear (no proof needed), and if it is linear write down its matrix.
   (a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$;
   (b) $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto 3x - 2y$;
   (c) $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto x^2 + y^2 + z^2$;
   (d) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x + 1, y + 1)$;
   (e) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x + y, 2x - 3y - z)$;
   (f) $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (3x, 7x)$;
   (g) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 3x + 5$;
   (h) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(x, y) \mapsto (3x - 5y, x + y, 4y)$.

(2) Find matrices for the following maps $\mathbb{R}^2 \to \mathbb{R}^2$ (you can assume that they are linear):
   (a) "Reflect in the y-axis".
   (b) "Reflect in the line $y = x$."
   (c) "Reflect in the line $y = 2x$." (hint: consider where $(1, 2)$ and $(-2, 1)$ go).
   (d) "Rotate by $\pi/3$ anticlockwise."
   (e) "Stretch by a factor of 3 in the x-direction."
   (f) "Rotate by $\pi/3$ anticlockwise and then reflect in the line $y = x$."
   (g) "Reflect in the line $y = x$ and then rotate by $\pi/3$ anticlockwise."

(3) Let $U = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$. Prove from the definition that $f : U \to \mathbb{R}$ defined by $f(x, y) = \frac{x}{y}$ is differentiable at $(1, 1)$ with derivative

$$(h, k) \mapsto h - k.$$

(4) Spivak 2-1.

(5) Show that the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = |x|$ is not differentiable at 0.

(6) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} \text{ if } (x, y) \neq (0, 0) \\ 0 \text{ if } (x, y) = (0, 0) \end{cases}$$

is continuous at 0 but is not differentiable there (hint: consider $f(h, h)$ as $h \to 0$ and $f(0, h)$ as $h \to 0$).

(7) Spivak 2-7.

(8) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}^2$, and $h : \mathbb{R}^2 \to \mathbb{R}$ be the linear maps defined by

$$\begin{aligned} f(x, y) &= (x + y, x - y, 2x + 3y) \\ g(x, y, z) &= (x - y - z, y + 2z) \\ h(x, y) &= x - y. \end{aligned}$$

Find the matrices of the following compositions (note: one of them does not exist!):

(a) $f \circ g$;
(b) $g \circ f$;
(c) $f \circ h$;
(d) $h \circ g \circ f$.

(9) Spivak 2-10, b,e,h,j.
(10) Spivak 2-11, a.
(11) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(r, \theta) = (r \cos \theta, r \sin \theta)$. Find Df and draw a picture illustrating your answer.