MATH 159 – HOMEWORK 6

Due 2pm on February 18th. Submit all problems; only the starred problems are graded. Throughout, F denotes a field.

(1) Let $V = \mathbb{R}^3$, a vector space over $\mathbb{R}$. Which of the following is a subspace of V?
   (a) $\{(x, y, z) \mid x + y + 2z = 0\}$.
   (b) $\{(x, y, z) \mid x + y + 2z = 1\}$.
   (c) $\{(x, y, z) \mid x + y + 2z = 0 \text{ and } x - y - 3z = 0\}$.
   (d) $\{(x, y, z) \mid x + y + 2z = 0 \text{ or } x - y - 3z = 0\}$.

(2) (*) Let V be a vector space over F. Prove that, if $\lambda \cdot v = 0$ for $\lambda \in F$ and $v \in V$, then $\lambda = 0$ or $v = \underline{0}$.

(3) (*) Let V be the set of functions $f : [0, 1] \to \mathbb{R}$, a vector space over $\mathbb{R}$. Which of the following is a subspace of V?
   (a) $\{f \mid f \text{ is bounded}\}$;
   (b) $\{f \mid |f| \text{ is bounded by } 1\}$;
   (c) $\{f \mid f \text{ is increasing}\}$;
   (d) $\{f \mid f(0) = 0\}$.

(4) (*) Show that any two vectors in F (regarded as an F-vector space) are linearly dependent. Show that any three vectors in F^2 are linearly dependent.

(5) Show that $\mathbb{R}$ is a vector space over $\mathbb{Q}$, with vector addition given by addition in $\mathbb{R}$ and scalar multiplication given by multiplication in $\mathbb{R}$. Show that $\{1, \sqrt{2}\}$ are linearly independent vectors in the $\mathbb{Q}$-vector space $\mathbb{R}$.

(6) (*) Let V be the vector space over F of sequences $(a_n)_{n\in\mathbb{N}}$ of elements of F. Show that

$$W = \{(a_n)_n \in V \mid a_{n+2} = a_{n+1} + a_n \text{ for all } n \in \mathbb{N}\ \}$$

is a subspace, and has a basis consisting of two vectors.

(7) Prove that a subset S of a vector space V over F is a basis if and only if every vector $v \in V$ can be written uniquely (up to reordering) in the form

$$v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \ldots + \alpha_m \cdot v_m$$

for some integer m, distinct vectors $v_1, \ldots, v_m \in S$ and non-zero elements $\alpha_1, \ldots, \alpha_m \in F$.

(8) (*) Show that if V is a vector space over F, $S \subset V$ is a spanning set, and $v \in V$ with $v \notin S$, then $S \cup \{v\}$ is linearly dependent.

(9) (*) Let $V = \mathbb{R}^3$, a vector space over $\mathbb{R}$. Find the span W of

$$\{(1, 2, 1), (3, -1, -4), (0, 7, 7)\}$$

in the form $\{(x, y, z) \in V \mid ax + by + cz = 0\}$ for some a, b, c. Find a basis for W.

(10) (optional) Let $\mathbb{Z}_2$ be the field with two elements, and let S be a finite set. The power set of S can be made into a vector space over $\mathbb{Z}_2$ by defining

$$A + B = (A \cup B) \setminus (A \cap B)$$

(the things in exactly one of A and B) for $A, B \subset S$ and defining scalar multiplication by $\lambda \cdot A = \emptyset$ if $\lambda = 0$ and A if $\lambda = 1$.

(a) Show that this is a vector space, and give a basis.

(b) Find the span of $\{A \subset S : |A| = 2\}$, and find a basis for this span.