MATH 163 – HOMEWORK 5

Due noon on May 8th.

(1) Prove that if $X \subset \mathbb{R}^n$ contains all of its accumulation points then X is closed *(hint: prove that if X^c is not open then it contains an accumulation point of X)*.

(2) Prove that, if $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U)$ is open for every open set $U \subset \mathbb{R}^m$, then f is continuous.

(3) Prove that $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact, without using the Heine–Borel theorem.

(4) Prove that any closed and bounded subset of $\mathbb{R}$ has a maximum and a minimum.

(5) The aim of this problem is to prove that, if $X \subset \mathbb{R}^n$ is compact, then X is closed and bounded. Suppose that X is compact.
   (a) For $r > 0$, let $U_r = \{x \in X : |x| < r\}$. Prove that each U_r is open, and that the U_r cover X.
   (b) Using the assumption that X is compact, deduce that X is bounded.
   (c) Let $y \in X^c$. For $\epsilon > 0$, let $U_\epsilon = \{x \in X : |y - x| > \epsilon\}$. Prove that each U_ϵ is open and that they cover X *(Make sure to say how you use the assumption that $y \in X^c$)*.
   (d) Deduce that some U_ϵ is contained in X^c and so that X is closed.

(6) Prove that xyz has a maximum value subject to the constraints $0 \leq x, y, z \leq 1$, $x + y + z = 1$. (optional) Find the maximum.

(7) Spivak 1-21.

(8) Let $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ and let $S' = S \setminus \{(1, 0, 0)\}$. Find a continuous surjective function $S' \to \mathbb{R}^2$. Is there a continuous surjective function $S \to \mathbb{R}^2$?

(9) (optional) Is there a continuous injective function $\mathbb{R}^2 \to \mathbb{R}$? (extremely hard) Is there a continuous surjective function $\mathbb{R} \to \mathbb{R}^2$?