MATH 159 – HOMEWORK 5

Due 2pm on February 10th. Submit all problems; only the starred problems are graded.

(1) (*) A sequence $(a_n)_n$ **tends to infinity** as $n \to \infty$ if, for every $X \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $a_n > X$ for all $n > N$. Show that, if a sequence is not bounded above, then it has a subsequence that tends to infinity.

(2) (*) Suppose that $(a_n)_n$ is a sequence of reals such that $\sum_{n=1}^{\infty} |a_n|$ converges. Show that $\sum_{n=1}^{\infty} a_n$ converges (hint: use the triangle inequality to show that the sequence of partial sums is Cauchy).

(3) Suppose that $(a_n)_n$ and $(b_n)_n$ are sequences of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, and $b_n \leq a_n$ for all n. Show that $\sum_{n=1}^{\infty} b_n$ converges.

(4) (*) Show that the series $\sum_{n=1}^{\infty} \frac{n}{4^n}$ converges. (optional) Find its limit.

(5) (*) For each of the following subsets of $\mathbb{R}$, say whether it is open, closed, both or neither (no proof required):
   (a) $\mathbb{R}$;
   (b) $[0, 1)$;
   (c) $\mathbb{Z}$;
   (d) $\mathbb{Q}$;
   (e) $\{x \in \mathbb{R} : x^3 \geq x\}$;
   (f) $\{x \in \mathbb{R} : x^3 > x\}$.

(6) Prove using de Morgan's law that any intersection of closed sets is closed, and that any finite union of closed sets is closed.

(7) Write $[0, 1)$ as a union of closed sets, and as an intersection of open sets.

(8) (*) Prove that, if $S \subset \mathbb{R}$ is closed and dense, then $S = \mathbb{R}$.

(9) (*) Use the contraction mapping theorem to show that $f(x) = 1 - \frac{x^3}{4}$ has a unique fixed point in $[0, 1]$.

(10) Find (with proof) an infinite closed subset of $[0, 1]$ that does not have any subset of the form $[a, b]$ for $a < b$.

(11) (optional) This question will guide you through the proof of the contraction mapping theorem. Let $S \subset \mathbb{R}$ be a closed subset, and let $f : S \to S$ be a function such that there is a positive constant $c < 1$ with

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in S$.

(a) Pick any $x_0 \in S$. Define a sequence $(x_n)_n$ by taking $x_{n+1} = f(x_n)$ for $n \geq 1$, so $x_n = f^n(x_0)$. Prove by induction that

$$|x_{n+1} - x_n| \leq c^n |x_1 - x_0|$$

for all $n \geq 1$.

(b) If $m \geq n$, by writing $x_m - x_n$ as

$$x_m - x_{m-1} + x_{m-1} - x_{m-2} + \ldots + x_{n+1} - x_n$$

and using part (a) and the triangle inequality, show that

$$|x_m - x_n| \leq \frac{c^n}{1-c}|x_1 - x_0|.$$

(c) Using (b), prove that $(x_n)_n$ is a Cauchy sequence, and therefore converges. Call its limit x.

(d) By writing $|f(x) - x| = |f(x) - f(x_n) + f(x_n) - x_n + x_n - x|$ and using the triangle inequality, prove that $f(x) = x$. (hint: show that for every $\epsilon > 0$, $|f(x) - x| < \epsilon$, by choosing an appropriate n.)

(e) Show that x is the unique fixed point of f (hint: suppose that y is another fixed point, and consider $|f(y) - f(x)|$).