MATH 159 – HOMEWORK 5

Due 1pm on November 4th. Submit all problems; only the starred problems are graded.

(1) (*)Which of the following sequences converges, and to what limit?
 (a) $a_n = \frac{1}{n^2+1}$ for $n \in \mathbb{N}$;
 (b) $a_n = \frac{n^2-1}{n^2+1}$ for $n \in \mathbb{N}$;
 (c) $a_n = \frac{2^n}{n}$ for $n \in \mathbb{N}$.

(2) (*)Let $a_n = (-1)^n + \frac{1}{n}$ for $n \in \mathbb{N}$. Find (with proof) $\lim \operatorname{lub} a_n$ and $\lim \operatorname{glb} a_n$.

(3) (*)Show that, if $0 < r < 1$, then $\lim_{n\to\infty} r^n = 0$ (hint: first show that the limit l exists, and then show that $l = rl$).

(4) (*)A sequence $(a_n)_n$ **tends to infinity** as $n \to \infty$ if, for every $X \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $a_n > X$ for all $n > N$. Show that, if $(a_n)_n$ is a sequence of positive real numbers, then either a_n contains a convergent subsequence, or a_n contains a subsequence that tends to infinity.

(5) Show that there is a sequence $(a_n)_n$ of real numbers such that, for every $x \in [0, 1]$, a_n has a subsequence converging to x.

(6) Suppose that $(a_n)_n$ and $(b_n)_n$ are sequences such that $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$. Show that $\sum_{n=1}^{\infty}(a_n + b_n)$ exists and is equal to $a + b$.

(7) (*)Suppose that $(a_n)_n$ is a sequence of reals such that $\sum_{n=1}^{\infty} |a_n|$ converges. Show that $\sum_{n=1}^{\infty} a_n$ converges (hint: use the triangle inequality).

(8) Suppose that $(a_n)_n$ and $(b_n)_n$ are sequences of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, and $b_n \le a_n$ for all n. Show that $\sum_{n=1}^{\infty} b_n$ converges.

(9) (*)Show that the series $\sum_{n=1}^{\infty} \frac{n}{4^n}$ converges.

(10) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

(11) (optional, harder) What is the value of $\sum_{n=1}^{\infty} \frac{n}{4^n}$?