# MATH 159 – HOMEWORK 4

Due 2pm on April 27th. Only problems with a * will be graded, but please submit solutions to everything. Whenever the domain of a function is not specified, it should be taken to be the largest possible subset of $\mathbb{R}$ on which the given definition makes sense.

(1) Determine whether each of the following limits exists, and if so, find it:
   (a) $\lim_{x\to 1} \frac{x^2-1}{x-1}$;
   (b) $\lim_{x\to 3} \frac{x+3}{x-3}$;
   (c) (*) $\lim_{x\to 2} \frac{x^3-8}{x-2}$;
   (d) (*) $\lim_{x\to 0} |x|$;
   (e) (*) $\lim_{x\to 0} \frac{\lfloor x\rfloor}{x}$;
   (f) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$.

(2) Figure out what "$\lim_{x\to\infty} f(x) = l$" should mean. Prove from your definition that $\lim_{x\to\infty} \frac{x}{x+1} = 1$.

(3) Figure out what "$\lim_{x\to a} f(x) = \infty$" should mean. Prove from your definition that $\lim_{x\to 0} \frac{1}{x^2} = \infty$. What about if $\frac{1}{x^2}$ is replaced by $\frac{1}{x}$?

(4) (*) Prove directly from the definitions (i.e. using $\epsilon$ and $\delta$) that the function $f : (0,\infty) \to (0,\infty)$ defined by $f(x) = 1/x$ is continuous at all $x \in (0,\infty)$.

(5) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x+1)^7$ is continuous.

(6) (*) Find functions $f, g : \mathbb{R} \to \mathbb{R}$ that are not continuous, but such that $f+g$ is continuous.

(7) Prove that, if $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous, then $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \max(f(x), g(x))$ is also continuous.

(8) (a) (*) Prove that a continuous, strictly increasing function $f : [a,b] \to [c,d]$ such that $f(a) = c$ and $f(b) = d$ is bijective.
   (b) Show that the function $f : [0,\infty) \to [0,\infty)$ given by $x \mapsto x^2$ is bijective.
   (c) Define $x \mapsto \sqrt{x}$ to be the inverse of the function $f$ from part b. Prove that $\sqrt{x}$ is continuous.

(9) Prove that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f^{-1}(U)$ is open for every open set $U \subset \mathbb{R}$. Is the converse true?

(10) (*) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed subset $X \subset \mathbb{R}$ such that $f(X)$ is not closed. What about if we also demand that $X$ be bounded?

(11) (*) For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, find all the points in $\mathbb{R}$ where the function is discontinuous (no proof required):
   (a) $f(x) = |x|/x$ if $x \neq 0$ and 0 if $x = 0$;
   (b) $f(x) = \lfloor x\rfloor$ where $\lfloor x\rfloor$ is the largest integer $\leq x$;
   (c) $f(x) = 1$ if $x$ is rational and 0 if $x$ is irrational.

(12) (optional) Find a function $f : [0,1] \to \mathbb{R}$ that is continuous at all irrational numbers, but discontinuous at all rational numbers (hint: let $f(x) = 0$ if $x$ is irrational, and something cunning if $x$ is rational).