MATH 159 – HOMEWORK 4

Due 2pm on February 3rd. Only problems with a * will be graded, but please submit solutions to everything.

(1) Prove that, if $a, b \in \mathbb{R}$, then $|ab| = |a||b|$.

(2) (*) Prove that, if $a, b \in \mathbb{R}$, then $||a| - |b|| \leq |a - b| \leq |a| + |b|$ (you can assume the triangle inequality).

(3) (*) Prove that, if $(a_n)_n$ and $(b_n)_n$ are sequences that converge to a and b respectively, then $(a_n + b_n)_n$ converges to $a + b$.

(4) (*) Give an example of two sequences $(a_n)_n$ and $(b_n)_n$ such that both of them are bounded, neither of them is convergent, but $(a_n + b_n)_n$ is convergent.

(5) (*) Which of the following sequences converges, and to what limit?
   (a) $a_n = \frac{1}{n^2+1}$ for $n \in \mathbb{N}$;
   (b) $a_n = \frac{n^2-1}{n^2+1}$ for $n \in \mathbb{N}$;
   (c) $a_n = \frac{2^n}{n}$ for $n \in \mathbb{N}$.

(6) (*) (Sandwich theorem) Prove that if $(a_n)_n, (b_n)_n, (c_n)_n$ are sequences such that a_n and c_n converge to the same limit a, and for all n

$$a_n \leq b_n \leq c_n,$$

then b_n also converges to a.

(7) Consider the sequence $(a_n)_n$ defined by $a_1 = 2$ and

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n}$$

for each $n \in \mathbb{N}$.
   (a) Prove that $a_n \geq \sqrt{2}$ for all n.
   (b) Prove that a_n is a bounded decreasing sequence.
   (c) Find, with proof, $\lim_{n\to\infty} a_n$.

(8) (*) Using the Archimedean property of $\mathbb{R}$, prove that $\bigcap_{n\in\mathbb{N}}(0, 1/n) = \emptyset$.

(9) (optional) Let $(a_{m,n})_{m,n}$ be a 'double sequence' of real numbers; that is, for every pair of integers $(m, n) \in \mathbb{N} \times \mathbb{N}$, $a_{m,n}$ is a real number. Suppose that, for every m, $(a_{m,n})_n$ converges to a limit x_m, and that for every n the sequence $(a_{m,n})_m$ converges to a limit y_n. [1]
   (a) Give an example in which $(x_m)_m$ and $(y_n)_n$ both converge, but converge to different limits. In other words,

$$\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} \neq \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}.$$

   (b) Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_{m,n} - x_m| < \epsilon$ and $|a_{m,n} - y_n| < \epsilon$ for every $m, n > N$. Show that, if $(x_m)_m$ and $(y_n)_n$ converge, then they have the same limit.

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[1] You can picture this as follows: arrange the $a_{m,n}$ in a grid, writing $a_{m,n}$ at the point with coordinates (m, n). Then the conditions say that the mth column converges to x_m and that the nth row converges to y_n.