MATH 159 – HOMEWORK 3

Due 2pm on April 20th. Only problems with a * will be graded, but please submit solutions to everything.

(1) (*) Prove that, if S and T are non-empty bounded subsets of $\mathbb{R}$, then $\text{lub}(S \cup T) = \max(\text{lub}(S), \text{lub}(T))$.

(2) Prove that, if $a, b \in \mathbb{R}$, then $||a| - |b|| \leq |a - b| \leq |a| + |b|$ (you can assume the triangle inequality).

(3) (*) Prove that if $(a_n)_n$ is a bounded sequence and $b_n \to 0$, then $a_n b_n \to 0$.

(4) (*) Give an example of two sequences $(a_n)_n$ and $(b_n)_n$ such that both of them are bounded, neither of them is convergent, but $(a_n + b_n)_n$ is convergent.

(5) (*) (Sandwich theorem) Prove that if $(a_n)_n, (b_n)_n, (c_n)_n$ are sequences such that a_n and c_n converge to the same limit a, and for all n

$$a_n \leq b_n \leq c_n,$$

then b_n also converges to a.

(6) (*) Which of the following sequences converges, and to what limit? (Give proofs! You can use the algebra of limits.)
   (a) $a_n = \frac{1}{n^2+1}$ for $n \in \mathbb{N}$;
   (b) $a_n = \frac{n^2-1}{n^2+1}$ for $n \in \mathbb{N}$;
   (c) $a_n = \frac{2^n}{n}$ for $n \in \mathbb{N}$.

(7) Consider the sequence $(a_n)_n$ defined by $a_1 = 2$ and

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n}$$

for each $n \in \mathbb{N}$.
   (a) Prove that $a_n \geq \sqrt{2}$ for all n.
   (b) Prove that a_n is a bounded decreasing sequence.
   (c) Find, with proof, $\lim_{n\to\infty} a_n$.

(8) (*) For each of the following subsets of $\mathbb{R}$, say whether it is open, closed, both or neither (no proof required):
   (a) $\mathbb{R}$;
   (b) $[0, 1)$;
   (c) $\{x \in \mathbb{R} : x^3 \geq x\}$;
   (d) $\{x \in \mathbb{R} : x^3 > x\}$.

(9) (*) Show that $\mathbb{Z}$ is a closed subset of $\mathbb{R}$, and that $\mathbb{Q}$ is neither closed nor open.

(10) Prove that, if $S \subset \mathbb{R}$ is closed and dense, then $S = \mathbb{R}$.

(11) Find an infinite closed subset of $[0, 1]$ that does not contain any non-empty open subset.