MATH 159 – HOMEWORK 3

Due 1pm on October 21st. Only problems with a * will be graded, but please submit solutions to everything.

(1) Suppose that a set $S \subset \mathbb{R}$ has a maximum element; that is, there is an an element $s \in S$ such that $s \geq t$ for all $t \in S$. Show that $\text{lub}(S) = s$.

(2) (*) Suppose that $S \subset \mathbb{R}$ and $T \subset \mathbb{R}$ are bounded below. Show directly from the definition of glb that:
   (a) $\text{glb}(S) = -\text{lub}(-S)$ where $-S = \{-s | s \in S\}$;
   (b) $\text{glb}(S \cup T) = \min\{\text{glb}(S), \text{glb}(T)\}$;
   (c) $\text{glb}(S + T) = \text{glb}(S) + \text{glb}(T)$ where $S + T = \{s + t | s \in S, t \in T\}$;
   (d) if c is positive, then $\text{glb}(c \cdot S) = c \cdot \text{glb}(S)$, where $c \cdot S = \{c \cdot s | s \in S\}$.

(3) (*) If S and T are bounded subsets of $\mathbb{R}$, and $S - T = \{s - t | s \in S, t \in T\}$, give a formula for $\text{lub}(S - T)$. Prove it (you can use the results of lectures and the previous question, provided that you state them clearly).

(4) (a) Show that, if a is an integer such that a^2 is divisible by three, then a is divisible by three.
   (b) Show that there is no $x \in \mathbb{Q}$ with $x^2 = 3$ (state clearly where you use the first part, if you do use it).
   (c) Show that there is an $x \in \mathbb{R}$ with $x^2 = 3$.

(5) (*)Show that the set of rationals whose denominator is a power of two is dense in $\mathbb{R}$; that is, show that for any $a, b \in \mathbb{R}$ with $a < b$ there exists an integer c and natural number n such that $a < \frac{c}{2^n} < b$.

(6) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function with the following properties:
   (a) f is strictly increasing; that is, if $a < b$ then $f(a) < f(b)$;
   (b) if $x \in \mathbb{Q}$, then $f(x) = x$.

   Show that $f(x) = x$ for all $x \in \mathbb{R}$. *(hint: use that $x = \text{lub}\{y \in \mathbb{Q} : y < x\}$ to show that $f(x) \geq x$; argue similarly with greatest lower bounds to show that $f(x) \leq x$.)*

(7) (*)A word is a finite string of letters from the usual 26 letter alphabet. Show that the set of words is countable.

(8) (*)Show that the set of subsets of $\mathbb{N}$ (that is, the power set of $\mathbb{N}$) is not countable. *(hint: suppose that $S_1, S_2, S_3, \ldots$ is a list of subsets of $\mathbb{N}$. Construct a subset that cannot be on this list.)*

(9) (optional, harder) Show that if S and T are sets, and if there are injective functions $f : S \to T$ and $g : T \to S$, then there is a bijection between S and T.