MATH 159 – HOMEWORK 2

Due 2pm on April 13th. Throughout, F denotes a field. Only problems with a * will be graded, but please submit solutions to everything.

(1) (*)From the field axioms show that, if $a, b \in F$, then $(-a) \times b = -(a \times b)$.

(2) From the field axioms show that, if $a, b \in F$ and $a \times b = 0$, then $a = 0$ or $b = 0$.

(3) (*)From the field axioms show that, if $a, b \in F$ and $a^2 = b^2$, then $a = b$ or $a = -b$.

(4) Review question 6 from the last homework, in which, for each natural number n, we defined the equivalence relation $\equiv_n$ on $\mathbb{Z}$. Let $\mathbb{Z}_n$ be the set of equivalence classes in $\mathbb{Z}$ under $\equiv_n$, and write $\overline{a}$ for the equivalence class of the integer a.

   (a) Think about why (6c) and (6d) from last time allow us to define operations $+$ and $\times$ on $\mathbb{Z}_n$ by the formulas $\overline{a} + \overline{b} = \overline{a + b}$ and $\overline{a} \times \overline{b} = \overline{a \times b}$.

   (b) (*) With these operations, $\mathbb{Z}_n$ satisfies the all the field axioms except possible the existence of multiplicative inverses.. Prove it.

   (c) (*) Which of the following are fields: $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$?

   (d) (optional) Make a conjecture about when $\mathbb{Z}_n$ is a field. Can you prove it?

(5) Let $F = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$, so $F \subset \mathbb{R}$. Show that F is a field (you can assume that $\mathbb{R}$ is a field, and that $\sqrt{2} \notin \mathbb{Q}$).

(6) Show from the order axioms that (for a, b in an ordered field):

   (a) (*)If $a < 0$ and $b < 0$ then $ab > 0$;

   (b) (*) $a^2 \geq 0$;

   (c) $2ab \leq a^2 + b^2$ with equality if and only if $a = b$.

   You may assume standard facts about fields, but must work directly from O1 - O4 (though you should feel free to prove any facts you need as lemmas!).

(7) Show that if a, b are non-zero elements of an ordered field, which are either both positive or both negative, and $a > b$, then $a^{-1} < b^{-1}$.

(8) (*) Show that, in an ordered field, there cannot be a smallest positive element (that is, if $a > 0$, then there exists $b > 0$ with $a > b$).

(9) (optional, harder) Find a field with four elements. Does there exist a field with six elements?