

MATH 159 – HOMEWORK 2

Due 1pm on January 20th. Only problems with a * will be graded, but please submit solutions to everything.

- (1) (*) From the ring axioms show that, if $a, b \in \mathbb{Z}$, then $(-a) \times (-b) = a \times b$.
- (2) (*) From the ring axioms together with cancellation show that, if $a, b \in \mathbb{Z}$ and $a^2 = b^2$, then $a = b$ or $a = -b$.
- (3) Review question 6 from the last homework, in which, for each natural number n , we defined the equivalence relation \equiv_n on \mathbb{Z} . Let \mathbb{Z}_n be the set of equivalence classes in \mathbb{Z} under \equiv_n , and write \bar{a} for the equivalence class of the integer a .
 - (a) Explain why (6c) and (6d) from last time allow us to define operations $+$ and \times on \mathbb{Z}_n by the formulas $\bar{a} + \bar{b} = \overline{a + b}$ and $\bar{a} \times \bar{b} = \overline{a \times b}$.
 - (b) (*) With these operations, \mathbb{Z}_n satisfies the ring axioms. Prove it.
 - (c) (*) Does \mathbb{Z}_6 satisfy cancellation? What about \mathbb{Z}_5 (hint: find multiplicative inverses)?
 - (d) Why does part (c) mean that cancellation cannot be deduced from the ring axioms?
- (4) Complete the missing proof from lectures that multiplication is well-defined on \mathbb{Q} .
- (5) Show from the order axioms that (for a, b in an ordered ring):
 - (a) (*) If $a < 0$ and $b < 0$ then $ab > 0$;
 - (b) $a^2 \geq 0$ and if $a^2 = 0$ then $a = 0$;
 - (c) (*) If $ab = 0$ then $a = 0$ or $b = 0$ (i.e. that cancellation holds in an ordered ring);
 - (d) $2ab \leq a^2 + b^2$.

You may assume standard facts about rings, but must work directly from O1 - O4 (though you should feel to prove any facts you need as lemmas!).

- (6) (*) Using the well-ordering principle, prove that, if $n \in \mathbb{N}$, then

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

- (7) Show that if a, b are non-zero elements of an ordered field, which are either both positive or both negative, and $a > b$, then $a^{-1} < b^{-1}$.
- (8) (*) Show that, in an ordered field, there cannot be a smallest positive element (that is, if $a > 0$, then there exists $b > 0$ with $a > b$).
- (9) (optional, harder) Find a set R with operations $+, \times$, satisfying all of the ring axioms except MC, in which multiplication is not commutative.