# MATH 159 – HOMEWORK 2

Due 1pm on October 14th. Only problems with a * will be graded, but please submit solutions to everything.

(1) (*)From the ring axioms show that, if $a, b \in \mathbb{Z}$, then $(-a) \times (-b) = a \times b$.

(2) (*)From the ring axioms together with cancellation show that, if $a, b \in \mathbb{Z}$ and $a^2 = b^2$, then $a = b$ or $a = -b$.

(3) Review question 6 from the last homework, in which, for each natural number $n$, we defined the equivalence relation $\equiv_n$ on $\mathbb{Z}$. Let $\mathbb{Z}_n$ be the set of equivalence classes in $\mathbb{Z}$ under $\equiv_n$, and write $\overline{a}$ for the equivalence class of the integer $a$.
   (a) Explain why (6c) and (6d) from last time allow us to define operations $+$ and $\times$ on $\mathbb{Z}_n$ by the formulas $\overline{a} + \overline{b} = \overline{a+b}$ and $\overline{a} \times \overline{b} = \overline{a \times b}$.
   (b) (*) With these operations, $\mathbb{Z}_n$ satisfies the ring axioms. Prove it.
   (c) (*) Does $\mathbb{Z}_6$ satisfy cancellation? What about $\mathbb{Z}_5$ (hint: find multiplicative inverses)?
   (d) Why does part (c) mean that cancellation cannot be deduced from the ring axioms?

(4) Complete the missing proof from lectures that addition is well-defined on $\mathbb{Q}$.

(5) Show from the order axioms that (for $a, b$ in an ordered ring):
   (a) (*)If $a < 0$ and $b < 0$ then $ab > 0$;
   (b) $a^2 \geq 0$ and if $a^2 = 0$ then $a = 0$;
   (c) (*)If $ab = 0$ then $a = 0$ or $b = 0$ (i.e. that cancellation holds in an ordered ring);
   (d) $2ab \leq a^2 + b^2$.

   You may assume standard facts about rings, but must work directly from O1 - O4 (though you should feel to prove any facts you need as lemmas!).

(6) (*)Using the well-ordering principle, prove that, if $n \in \mathbb{N}$, then

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

(7) Show that if $a, b$ are non-zero elements of an ordered field, which are either both positive or both negative, and $a > b$, then $a^{-1} < b^{-1}$.

(8) (*) Show that, in an ordered field, there cannot be a smallest positive element (that is, if $a > 0$, then there exists $b > 0$ with $a > b$).

(9) (optional, harder) Find a set $R$ with operations $+, \times$, satisfying all of the ring axioms except MC, in which multiplication is not commutative.