MATH 159 – HOMEWORK 1

Due 2pm on January 13th. In all questions, A, B and C denote sets contained in some 'universal' set X. Only problems with a * will be graded, but please submit solutions to everything.

(1) (*) Show that if $A \subset B$ and $B \subset C$, then $A \subset C$.
(2) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(3) (*) Prove the first of de Morgan's laws: $(A \cup B)^c = A^c \cap B^c$.
(4) If S is a set, then the **power set** $P(S)$ is defined to be the set of all subsets of S. That is,
$$P(S) = \{A | A \subset S\}.$$
  (a) (*)Write down the power set of $\{1, 2, 3\}$.
  (b) (*)What is the power set of $\emptyset$?
  (c) Construct a bijection between $P(S)$ and the set of functions $f : S \to \{0, 1\}$.
(5) Suppose that $\sim$ is a relation on a set S that is symmetric, transitive, and has the following property: if $a \in S$ then there exists $b \in S$ such that $a \sim b$. Show that $\sim$ is an equivalence relation.
(6) In lectures we defined the equivalence relation $\equiv_n$ on $\mathbb{Z}$, for any natural number n: $a \equiv_n b$ if and only if $n | b - a$; that is, there exists an integer m with $(b - a) = nm$.
  (a) (*) Show that this is indeed an equivalence relation.
  (b) How many equivalence classes are there? What is the equivalence class containing 0?
  (c) (*) Show that, if $a \equiv_n a'$ and $b \equiv_n b'$, then $a + b \equiv_n a' + b'$.
  (d) Show that, if $a \equiv_n a'$ and $b \equiv_n b'$, then $ab \equiv_n a'b'$.
(7) Prove that, if $f : A \to B$ and $g : B \to C$ are both surjective functions, then $g \circ f$ is also surjective.
(8) (*) Prove that, if $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is injective, then f is injective. Give an example to show that if $g \circ f$ and f are both injective, then g need not be injective.
(9) Let $f : A \to B$ be a function. If $A_1 \subset A$, define
$$f(A_1) = \{f(a) | a \in A_1\}.$$
If $B_1 \subset B$, define
$$f^{-1}(B_1) = \{a \in A | f(a) \in B_1\}.$$
These are respectively called the image of A_1 and pre-image of B_1.
  (a) (*) Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
  (b) (*) Show that $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ but give an example to show that this need not be an equality.
  (c) What about $f^{-1}(B_1 \cup B_2)$?
(10) (optional) Suppose that A and B are finite sets. How many relations are there on A? How many functions $f : A \to B$ are there? How many injective

functions? How many bijections? How many equivalence relations? How many surjective functions? (note: the last two parts are extremely difficult!)