THE BREUIL–MÉZARD CONJECTURE WHEN $l \neq p$.

ABSTRACT. Let l and p be primes, let $F/\mathbb{Q}_p$ be a finite extension with absolute Galois group G_F, let $\mathbb{F}$ be a finite field of characteristic l, and let

$$\overline{\rho} : G_F \to GL_n(\mathbb{F})$$

be a continuous representation. Let $R^{\square}(\overline{\rho})$ be the universal framed deformation ring for $\overline{\rho}$. If $l = p$, then the Breuil–Mézard conjecture (as formulated in [EG14]) relates the mod l reduction of certain cycles in $R^{\square}(\overline{\rho})$ to the mod l reduction of certain representations of $GL_n(\mathcal{O}_F)$. We state an analogue of the Breuil–Mézard conjecture when $l \neq p$, and prove it whenever $l > 2$ using automorphy lifting theorems. We give a local proof when l is "quasi-banal" for F and $\overline{\rho}$ is tamely ramified. We also analyse the reduction modulo l of the types $\sigma(\tau)$ defined by Schneider and Zink [SZ99].

CONTENTS

1. INTRODUCTION

When F is a p-adic field and $\overline{\rho}$ is an n-dimensional mod p representation of its absolute Galois group G_F, the Breuil–Mézard conjecture relates singularities in the deformation ring of $\overline{\rho}$ to the mod p representation theory of $GL_n(\mathcal{O}_F)$. It was first formulated, for $F = \mathbb{Q}_p$ and $n = 2$, in [BM02], and (mostly) proved in this case in [Kis09a]. In full generality, the conjecture is formulated in [EG14] but is not known in any cases with $n > 2$. In this article we prove an analogue of the Breuil–Mézard conjecture for mod l representations of G_F and $GL_n(\mathcal{O}_F)$, with F a p-adic field and l an odd prime *distinct* from p.

We give a precise statement, after setting up a little notation. Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$, residue field k_F of order q, and absolute Galois group G_F, and let l be a prime distinct from p. Let E be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}$, uniformiser λ, and residue field $\mathbb{F}$. Let

$$\overline{\rho} : G_F \to GL_n(\mathbb{F})$$

be a continuous representation. Then there is a universal framed deformation ring $R^{\square}(\overline{\rho})$ parameterizing lifts of $\overline{\rho}$. Our main result, stated below, relates congruences

between irreducible components of $\operatorname{Spec} R^{\square}(\bar{\rho})$ to congruences between representations of $GL_n(\mathcal{O}_F)$.

It is known that $\operatorname{Spec} R^{\square}(\bar{\rho})$ is flat and equidimensional of relative dimension n^2 over $\operatorname{Spec} \mathcal{O}$ – see Theorem 2.5. Let $\mathcal{Z}(R^{\square}(\bar{\rho}))$ be the free abelian group on the irreducible components[1] of $\operatorname{Spec} R^{\square}(\bar{\rho})$; similarly we have the group $\mathcal{Z}(\overline{R}^{\square}(\bar{\rho}))$ where $\overline{R}^{\square}(\bar{\rho}) = R^{\square}(\bar{\rho}) \otimes_{\mathcal{O}} \mathbb{F}$. There is a natural homomorphism

$$\operatorname{red} : \mathcal{Z}(R^{\square}(\bar{\rho})) \longrightarrow \mathcal{Z}(\overline{R}^{\square}(\bar{\rho}))$$

taking an irreducible component of $\operatorname{Spec}(R^{\square}(\bar{\rho}))$ to its intersection with the special fibre (counted with multiplicities).

An **inertial type** is an isomorphism class of continuous representation $\tau : I_F \to GL_n(\overline{E})$ that may be extended to G_F. If τ is an inertial type, then there is a quotient $R^{\square}(\bar{\rho}, \tau)$ of $R^{\square}(\bar{\rho})$ that (roughly speaking) parameterizes representations of type τ; that is, whose restriction to I_F is isomorphic to τ. Then $\operatorname{Spec} R^{\square}(\bar{\rho}, \tau)$ is a union of irreducible components of $\operatorname{Spec} R^{\square}(\bar{\rho})$.

Let $R_E(GL_n(\mathcal{O}_F))$ (resp. $R_{\mathbb{F}}(GL_n(\mathcal{O}_F))$) be the Grothendieck group of finite dimensional smooth representations of $GL_n(\mathcal{O}_F)$ over E (resp. $\mathbb{F}$), and let

$$\operatorname{red} : R_E(GL_n(\mathcal{O}_F)) \to R_{\mathbb{F}}(GL_n(\mathcal{O}_F))$$

be the surjective map given by reducing a representation modulo l. In section 4 we define a homomorphism

$$\operatorname{cyc} : R_E(GL_n(\mathcal{O}_F)) \to \mathcal{Z}(R^{\square}(\bar{\rho}))$$

by the formula

$$\operatorname{cyc}(\theta) = \sum_{\tau} m(\theta^{\vee}, \tau) Z(R^{\square}(\bar{\rho}, \tau))$$

where the sum is over all inertial types, $Z(R^{\square}(\bar{\rho}, \tau))$ is the formal sum of the irreducible components of $\operatorname{Spec} R^{\square}(\bar{\rho}, \tau)$, and $m(\theta^{\vee}, \tau)$ is the multiplicity of $\theta^{\vee}$ in any generic irreducible admissible representation π such that $\operatorname{rec}_l(\pi)|_{I_F} \cong \tau$.

Theorem. *Suppose that $l > 2$. There is a unique map $\overline{\operatorname{cyc}}$ making the following diagram commute:*

(1)
$$\begin{array}{ccc} R_E(GL_n(\mathcal{O}_F)) & \xrightarrow{\operatorname{cyc}} & \mathcal{Z}(R^{\square}(\bar{\rho})) \\ {\scriptstyle\operatorname{red}}\downarrow & & {\scriptstyle\operatorname{red}}\downarrow \\ R_{\mathbb{F}}(GL_n(\mathcal{O}_F)) & \xrightarrow{\overline{\operatorname{cyc}}} & \mathcal{Z}(\overline{R}^{\square}(\bar{\rho})). \end{array}$$

This is Theorem 4.6 below. We conjecture (Conjecture 4.5) that it is also true for $l = 2$. The content of the theorem is that congruences between representations of $GL_n(\mathcal{O}_F)$ force congruences between irreducible components of $R^{\square}(\bar{\rho})$.

The image of the map cyc is precisely the $\mathbb{Z}$-span of the cycles $Z(R^{\square}(\bar{\rho}, \tau))$. By inverting cyc one obtains a formula for the cycles $\operatorname{red}(Z(R^{\square}(\bar{\rho}, \tau)))$ in terms of the map $\overline{\operatorname{cyc}}$, and hence for the Hilbert–Samuel multiplicities of the special fibres of these fixed-type deformation rings. The map cyc also appears in the $l = p$ situation when working with potentially semistable (rather than potentially crystalline) deformation rings.

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[1]We suppose that E is "sufficiently large" and in particular that all of these are geometrically irreducible.

Our proof of this theorem is 'global', making use of the methods of [GK14] and [EG14]. We use the Taylor–Wiles–Kisin patching method to produce an exact functor

$$\theta \longmapsto H_\infty(\theta)$$

from the category of finitely generated $\mathcal{O}$-modules with a smooth $GL_n(\mathcal{O}_F)$-action to the category of finitely generated $R^\square(\overline{\rho})$-modules, such that the support of $H_\infty(\theta)$ – counted with multiplicity – is $\mathrm{cyc}(\theta)$. As this functor is compatible with reduction modulo l, we can deduce the theorem.

We can also give local proofs of (variants of) the theorem in some special cases. In [Sho13], we studied the case $n = 2$ and $l > 2$; explicitly calculated the rings $R^\square(\overline{\rho}, \tau)$ in this case and gave a local proof of the theorem.[2] In section 7 we prove the theorem (with $R_E(GL_n(\mathcal{O}_F))$ replaced by a certain subgroup of $R_E(GL_n(k_F))$) in the case that $\overline{\rho}$ is tamely ramified and l is *quasi-banal*; that is, $l > n$ and $l | q - 1$. The method is to first observe that there is a scheme $\mathfrak{X}$ of finite type over $\mathrm{Spec}\,\mathcal{O}$ — it is the moduli space of pairs of invertible matrices Σ and Φ satisfying $\Phi\Sigma\Phi^{-1} = \Sigma^q$ — such that the $\mathrm{Spf}\,R^\square(\overline{\rho})$, for varying $\overline{\rho}$, may all be obtained as the completions of $\mathfrak{X}$ at closed points. This allows us to reduce the theorem to the case in which $\overline{\rho}$ is "distinguished"; this is a certain genericity condition. When $\overline{\rho}$ is distinguished we can compute all of the $R^\square(\overline{\rho}, \tau)$ by elementary arguments. As we also have a good understanding of the representation theory of $GL_n(k_F)$ in the quasi-banal case, we can deduce the theorem. It seems likely that these methods could be pushed further; we have just dealt with the simplest interesting case for general n.

Kisin [Kis09a] proved most cases of the original Breuil–Mézard conjecture, simultaneously with proving most cases of the Fontaine–Mazur conjecture for $GL_2/\mathbb{Q}$. The point is that the information about the special fibres of local deformation rings provided by the Breuil–Mézard conjecture is what is needed to prove automorphy lifting theorems in general weight, using the Taylor–Wiles method as modified by Kisin in [Kis09b]. The methods of [GK14], [EG14] and this article can be viewed as implementing this idea "in reverse", using known automorphy lifting theorems (or, in the case of [EG14], assuming automorphy lifting theorems) to deduce the Breuil–Mézard conjecture. We note, however, that no cases of the Breuil–Mézard conjecture are known when $l = p$ and $n > 2$, the question being bound up with the weight part of Serre's conjecture and the Fontaine–Mazur conjecture.

The other motivation behind our theorem is the "Ihara avoidance" method of [Tay08], which is in the $l \neq p$ setting. Taylor's idea is to compare the special fibres of very specific $R^\square(\overline{\rho}, \tau)$, and combine this with the Taylor–Wiles–Kisin method to prove non-minimal automorphy lifting theorems (i.e. automorphy lifting theorems incorporating a change of level). The similarity to Kisin's use of the Breuil–Mézard conjecture to prove automorphy lifting theorems with a change of weight is clear; thus it is natural to try to study local deformation rings when $l \neq p$ from the point of view of the Breuil–Mézard conjecture. Our proof actually *depends* on Taylor's results, as it makes crucial use of non-minimal automorphy lifting theorems.

Section 6 has a rather different focus. Certain of the representations of $GL_n(\mathcal{O}_F)$ are more interesting than the others; these are the K-types. For every inertial type τ there is a corresponding K-type $\sigma(\tau)$, essentially constructed by Schneider and Zink [SZ99]. These representations have an interesting 'Galois theoretic' interpretation

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[2] Strictly speaking, this proof works with $R_E(GL_n(\mathcal{O}_F))$ replaced by the subgroup generated by 'K-types'.

— see Theorem 3.7 below. We determine the multiplicities $m(\sigma(\tau), \tau')$ when τ and τ' are inertial types; the answer is given in terms of certain Kostka numbers. We also explain how to determine the mod l reduction of the representations $\sigma(\tau)$ in terms of the mod l reduction of representations of certain general linear groups; in order to do this, we must work with a variant of the construction of [SZ99]. Sections 6.1 to 6.3 are used in section 7, but otherwise the only place that section 6 is used in the rest of the paper is to derive the multiplicity formula of Proposition 4.3; in particular, sections 6 and 7 are not required for the proof of Theorem 4.6.

1.1. **Acknowledgements.**

2. Deformation rings

2.1. **Definitions.** Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_F$, and residue field k_F of order q. Let $\overline{F}$ be an algebraic closure of F, $\overline{k}_F$ the induced algebraic closure of k_F, and $G_F = \mathrm{Gal}(\overline{F}/F)$. Let $I_F \lhd G_F$ and $P_F \lhd G_F$ be, respectively, the inertia and wild inertia subgroups of G_F. We have canonical isomorphisms

$$G_F/I_F = \hat{\mathbb{Z}}$$

and

$$I_F/P_F = \varprojlim_{k/k_F} k^\times \cong \prod_{l \neq p} \mathbb{Z}_l(1),$$

where the limit is over finite extensions of k_F contained in $\overline{k}_F$ and the transition maps are the norm maps. Let $\phi \in G_F/I_F$ be arithmetic Frobenius, and denote also by ϕ a choice of lift to G_F. Let σ be a topological generator for I_F/P_F; this choice is equivalent to choosing a norm-compatible system of generators for the units in each finite extension k of k_F, or to choosing a basis for each $\mathbb{Z}_l(1)$. Then, via these choices, G_F/P_F is isomorphic to the profinite completion of

$$(2) \qquad \langle \phi, \sigma | \phi\sigma\phi^{-1} = \sigma^q \rangle = \mathbb{Z} \ltimes \mathbb{Z}[\tfrac{1}{p}].$$

Let $E/\mathbb{Q}_l$ be a finite extension with ring of integers $\mathcal{O}$, uniformiser λ and residue field $\mathbb{F}$. Let $\mathcal{C}_\mathcal{O}$ denote the category of artinian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$, and $\mathcal{C}_\mathcal{O}^\wedge$ the category of complete noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$. If A is an object of $\mathcal{C}_\mathcal{O}$ or $\mathcal{C}_\mathcal{O}^\wedge$, let $\mathfrak{m}_A$ be its maximal ideal.

Suppose that $\overline{M}$ is an n-dimensional $\mathbb{F}$-vector space and that $\overline{\rho} : G_F \to \mathrm{Aut}_\mathbb{F}(\overline{M})$ is a continuous homomorphism. Let $(\overline{e}_i)_{i=1}^n$ be a basis for $\overline{M}$, so that $\overline{\rho}$ gives a map $\overline{\rho} : G_F \to GL_n(\mathbb{F})$.

Define two functors

$$D(\overline{\rho}), D^\square(\overline{\rho}) : \mathcal{C}_\mathcal{O} \to \mathbf{Set}$$

as follows:

- $D(\overline{\rho})(A)$ is the set of equivalence classes of (M, ρ, ι) where: M is a free rank n A-module, $\rho : G_F \to \mathrm{Aut}_A(M)$ is a continuous homomorphism, and

$$\iota : M \otimes_A \mathbb{F} \xrightarrow{\sim} \overline{M}$$

is an isomorphism commuting with the actions of G_F;
- $D^\square(\overline{\rho})(A)$ is the set of equivalence classes of $(M, \rho, (e_i)_{i=1}^n)$ where: M is a free rank n A-module, $\rho : G_F \to \mathrm{Aut}_A(M)$ is a continuous homomorphism, and $(e_i)_{i=1}^n$ is a basis of M such that the isomorphism $\iota : M \otimes_A \mathbb{F} \xrightarrow{\sim} \overline{M}$ taking $e_i \otimes 1$ to $\overline{e}_i$ commutes with the actions of G_F.

In the first case, (M, ρ, ι) and (M', ρ', ι') are equivalent if there is an isomorphism $\alpha : M \to M'$, commuting with the actions of G_F, such that $\iota = \iota' \circ \alpha$; in the second case, $(M, \rho, (e_i)_i)$ and $(M', \rho', (e'_i)_i)$ are equivalent if the isomorphism of A-modules $M \to M'$ defined by $e_i \mapsto e'_i$ commutes with the actions of G_F. There is a natural transformation of functors $D^\square(\overline{\rho}) \to D(\overline{\rho})$ given by forgetting the basis.

Alternatively, when $\overline{\rho}$ is regarded as a homomorphism to $GL_n(\mathbb{F})$, we have the equivalent definitions

$$D^\square(\overline{\rho})(A) = \{\text{continuous } \rho : G_F \to GL_n(A) \text{ lifting } \overline{\rho}\}$$

and

$$D(\overline{\rho})(A) = \{\text{continuous } \rho : G_F \to GL_n(A) \text{ lifting } \overline{\rho}\}/1 + M_n(\mathfrak{m}_A)$$

where the action of the group $1 + M_n(\mathfrak{m}_A)$ is by conjugation.

The functor $D(\overline{\rho})$ is not usually pro-representable, but the functor $D^\square(\overline{\rho})$ always is (see, for example, [Kis09b] (2.3.4)):

Definition 2.1. The *universal lifting ring* (or universal framed deformation ring) of $\overline{\rho}$ is the object $R^\square(\overline{\rho})$ of $\mathcal{C}_\mathcal{O}^\wedge$ that pro-represents the functor $D^\square(\overline{\rho})$. The universal lift is denoted $\rho^\square : G_F \to GL_n(R^\square(\overline{\rho}))$.

2.2. **Geometry of $R^\square(\overline{\rho})$.** Recall the following calculation from [BLGGT14] §1.2:

Lemma 2.2. *The scheme* $\mathrm{Spec}\, R^\square(\overline{\rho})[1/l]$ *is generically formally smooth of dimension n^2.* $\qquad\square$

Let $I_F \to I_F/\tilde{P}_F$ be the maximal pro-l quotient of I_F. The next lemma enables us to reduce to the case where the residual representation is trivial on $\tilde{P}_F$. Suppose that θ is an irreducible $\mathbb{F}$-representation of $\tilde{P}_F$; write $[\theta]$ for the orbit of the isomorphism class of θ under conjugation by G_F. By [CHT08] Lemma 2.4.11, θ may be extended to an $\mathcal{O}$-representation $\tilde{\theta}$ of G_θ where G_θ is the open subgroup $\{g \in G_F : g\theta g^{-1} \cong \theta\}$ of G_F. For each irreducible representation θ of $\tilde{P}_F$, we pick such a $\tilde{\theta}$. If M is a finite-dimensional $\mathbb{F}$-vector space with a continuous action of G_F, then define

$$M_\theta = \mathrm{Hom}_{\tilde{P}_F}(\tilde{\theta}, M).$$

This has a natural continuous action of G_θ given by $(gf)(v) = gf(g^{-1}v)$; the subgroup $\tilde{P}_F$ of G_θ acts trivially. If $\overline{\rho} : G_F \to GL_n(\mathbb{F})$ is continuous and corresponds to some choice of basis for M, then choose a basis for each M_θ to obtain a continuous homomorphism $\overline{\rho}_\theta : G_\theta \to GL_n(\mathbb{F})$.

Lemma 2.3. *(Tame reduction) If $R^\square(\overline{\rho}_\theta)$ is the universal framed deformation ring for the representation $\overline{\rho}_\theta$ of $G_\theta/\tilde{P}_F$, then*

$$R^\square(\overline{\rho}) \cong \left(\widehat{\bigotimes}_{[\theta]} R^\square(\overline{\rho}_\theta)\right)[[X_1, \ldots, X_{n^2 - \sum n_\theta^2}]]$$

where $n_\theta = \dim \overline{\rho}_\theta$.

Proof. This is a modification, due to Choi [Cho09], of [CHT08] Corollary 2.4.13 to take into account the framings. See [Sho13] Lemma 2.3. $\square$

The next result is due to David Helm, and will appear in a forthcoming paper of his. I thank him for allowing me to include the proof here.

Definition 2.4. Suppose that R is a ring. Let $\mathcal{M}(n, q)_R$ be the moduli space (over $\operatorname{Spec} R$) of pairs of matrices $\Sigma, \Phi \in GL_{n,R} \times_{\operatorname{Spec} R} GL_{n,R}$ such that

$$\Phi \Sigma \Phi^{-1} = \Sigma^q.$$

It is the closed subscheme of $GL_{n,R} \times_{\operatorname{Spec} R} GL_{n,R}$ cut out by the n^2 matrix coefficients of the above equation. Denote by π_Σ the morphism

$$\pi_\Sigma : \mathcal{M}(n, q)_R \longrightarrow GL_{n,R}$$

$$(\Sigma, \Phi) \mapsto \Sigma.$$

Theorem 2.5. *The scheme* $\operatorname{Spec} R^\square(\overline{\rho})$ *is a reduced complete intersection, flat and equidimensional of relative dimension* n^2 *over* $\operatorname{Spec} \mathcal{O}$.

Proof. Suppose that k is an algebraically closed field of characteristic distinct from p, and consider $\mathcal{M}(n, q)_k$. Let Σ_0 be a closed point in the image of π_Σ, let Z_0 be the centraliser of Σ_0 in $GL_{n,k}$ (a closed subgroup scheme of $GL_{n,k}$) and let C_0 be the conjugacy class of Σ_0 in $GL_{n,k}$, a locally closed subscheme of $GL_{n,k}$ isomorphic to $GL_{n,k}/Z_0$. Then $\pi_\Sigma^{-1}(\Sigma_0)$ is (by right multiplication on Φ) a Z_0-torsor. Thus the preimage $\pi_\Sigma^{-1}(C_0)$ in $\mathcal{M}(n, q)_k$ has dimension

$$\dim C_0 + \dim Z_0 = n^2 - \dim Z_0 + \dim Z_0 = n^2.$$

Since the eigenvalues of any Σ in the image of π_Σ must be $(q^n - 1)$th roots of unity, the number of conjugacy classes C_0 of matrices in the image of π_Σ is finite.[3] Therefore

$$\dim \mathcal{M}(n, q)_k = n^2.$$

Now let $R = \mathcal{O}$. We see that $\mathcal{M}(n, q)_\mathcal{O} \to \operatorname{Spec} \mathcal{O}$ is equidimensional of dimension n^2. But the smooth scheme $GL_{n,\mathcal{O}} \times_\mathcal{O} GL_{n,\mathcal{O}}$ has relative dimension $2n^2$ over $\operatorname{Spec} \mathcal{O}$ and $\mathcal{M}(n, q)_\mathcal{O}$ is a closed subscheme cut out by n^2 equations; it follows that $\mathcal{M}(n, q)_\mathcal{O}$ is a local complete intersection. In particular, it is a Cohen–Macaulay scheme. As its fibres over the regular local ring $\operatorname{Spec} \mathcal{O}$ are of the same dimension, n^2, it is flat over $\operatorname{Spec} \mathcal{O}$.

Now, by Lemma 2.3, the assertions of the theorem may be reduced to the case in which $\overline{\rho}$ is tamely ramified (using Lemma 3.3 of [BLGHT11] to propagate flatness, reducedness, and dimension from objects of $\mathcal{C}_\mathcal{O}^\wedge$ to their completed tensor products). In this case, any lift of $\overline{\rho}$ to an object of $\mathcal{C}_\mathcal{O}$ is also tamely ramified, as P_F is pro-p. Our choice of topological generators ϕ and σ for G_F/P_F satisfying the equation $\phi \sigma \phi^{-1} = \sigma^q$ provides a closed point of $\mathcal{M}(n, q)_\mathcal{O}$ corresponding to $\overline{\rho}$ and identifies $R^\square(\overline{\rho})$ with the completion of the local ring of $\mathcal{M}(n, q)_\mathcal{O}$ at this point (to see this, compare the A-valued points for A an object of $\mathcal{C}_\mathcal{O}$). Therefore, by the corresponding facts for $\mathcal{M}(n, q)_\mathcal{O}$, we have shown that $R^\square(\overline{\rho})$ is a complete

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[3] Here we use that $q > 1$. It is unknown whether the moduli space of pairs of commuting matrices over $\mathbb{C}$ is Cohen–Macaulay (or even reduced!), although this is conjectured to be the case (see [Hai99]).

intersection and is flat over $\mathcal{O}$. It is reduced since it is generically reduced (by Lemma 2.2 and the fact that it is $\mathcal{O}$-flat) and Cohen–Macaulay. $\qquad\square$

We extract the following consequence of the proof:

Proposition 2.6. *If k is a field of characteristic distinct from p that contains all of the $(q^n - 1)$th roots of unity, and C is a conjugacy class in $GL_n(\overline{k})$ that is stable under the qth power map, then the Zariski closure in $\mathcal{M}(n, q)_k$ of $\pi_{\Sigma}^{-1}(C)$ is an absolutely irreducible component of $\mathcal{M}(n, q)_k$, denoted $\mathcal{M}(n, q, \Sigma \sim C)_k$. Every irreducible component of $\mathcal{M}(n, q)_k$ is of this form.*

Proof. As C is stable under the qth power map and k contains the $(q^n - 1)$th roots of unity, C contains a k-point. Then C is absolutely irreducible and the fibres of π_{Σ} above points of C are all absolutely irreducible of dimension $\dim \mathcal{M}(n, q)_k - \dim C$. Therefore the closure of $\pi_{\Sigma}^{-1}(C)$ is absolutely irreducible of the same dimension as $\dim \mathcal{M}(n, q)_k$, and is therefore an absolutely irreducible component.

As every point of $\mathcal{M}(n, q)_k$ is in $\pi_{\Sigma}^{-1}(C)$ for some C, we obtain the final statement. $\qquad\square$

2.3. **Cycles.** Suppose that X is a noetherian scheme and that $\mathcal{F}$ is a coherent sheaf on X. Let Y be the scheme-theoretic support of $\mathcal{F}$, and let $d \geq \dim Y$. Let $\mathcal{Z}^d(X)$ be the free abelian group on the d-dimensional points of X; elements of $\mathcal{Z}^d(X)$ are called d-dimensional cycles. If $\mathfrak{a} \in X$ is a point of dimension d write $[\mathfrak{a}]$ for the corresponding element of $\mathcal{Z}^d(X)$ and define the multiplicity $e(\mathcal{F}, \mathfrak{a})$ to be the length of $\mathcal{F}_{\mathfrak{a}}$ as an $\mathcal{O}_{Y, \mathfrak{a}}$-module (this is zero if $\mathfrak{a} \notin Y$).

Definition 2.7. The cycle $Z^d(\mathcal{F})$ associated to $\mathcal{F}$ is the element
$$\sum_{\mathfrak{a}} e(\mathcal{F}, \mathfrak{a})[\mathfrak{a}] \in \mathcal{Z}^d(X).$$

If $X = \operatorname{Spec} A$ is affine and $\mathcal{F} = \widetilde{M}$ is the coherent sheaf associate to a finitely generated A-module M, then we will write $Z^d(M)$ for $Z^d(\mathcal{F})$. If X is equidimensional of dimension d, then we will usually drop d from the notation, so that $\mathcal{Z}(X) = \mathcal{Z}^d(X)$, $Z(\mathcal{F}) = Z^d(\mathcal{F})$ etc.

If $i : X \to X'$ is a closed immersion of X in a noetherian scheme X', then there is a natural inclusion $i_* : \mathcal{Z}^d(X) \to \mathcal{Z}^d(X')$ for each d. For a coherent sheaf $\mathcal{F}$ on X whose support has dimension at most d, we then have
$$i_*(Z^d(\mathcal{F})) = Z^d(i_*(\mathcal{F})).$$
We will often use this compatibility without comment.

If X is a noetherian scheme of dimension d and $X \to \operatorname{Spec} \mathcal{O}$ is a flat morphism, then let
$$j : \overline{X} = X \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathbb{F} \to X$$
be the inclusion of the special fibre and denote by red the reduction map
$$\operatorname{red} : \mathcal{Z}(X) \to \mathcal{Z}(\overline{X})$$
which takes a d-dimensional point $\mathfrak{a}$ with closure Y to the cycle $Z^{d-1}(j^* \mathcal{O}_Y)$. The following is a special case of [EG14] Proposition 2.2.13:

Lemma 2.8. *In the above situation, if $\mathcal{F}$ is a coherent sheaf on X such that multiplication by λ is injective on $\mathcal{F}$, then*
$$\operatorname{red}(Z^d(\mathcal{F})) = Z^d(j^*(\mathcal{F})).$$
$\qquad\square$

If $f : X \to Y$ is a flat morphism of noetherian schemes, with X and Y equidimensional of dimensions d and e respectively, then we define a map

$$f^* : \mathcal{Z}^e(Y) \to \mathcal{Z}^d(X)$$

by taking a point $\mathfrak{a} \in Y$ with closure Z of dimension e to the cycle

$$Z^d(f^*\mathcal{O}_Z) \in \mathcal{Z}^d(X).$$

Lemma 2.9. *In the above situation, if $\mathcal{F}$ is a coherent sheaf on Y then*

$$f^*(Z^d(\mathcal{F})) = Z^d(f^*(\mathcal{F})).$$

Proof. We may suppose that $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$ for noetherian rings R and S, so that f induces a flat map $f^* : R \to S$, and $\mathcal{F} = \widetilde{M}$ for a finitely generated R-module M. If $\mathfrak{b}$ is a minimal prime of S and $\mathfrak{a} = \mathfrak{b} \cap R$, then $\mathfrak{a}$ is a minimal prime of R (by the going down property of flat morphisms) and we must show:

$$\operatorname{length}_{R_{\mathfrak{a}}}(M_{\mathfrak{a}}) \operatorname{length}_{S_{\mathfrak{b}}}((R/\mathfrak{a} \otimes_R S)_{\mathfrak{b}}) = \operatorname{length}_{S_{\mathfrak{b}}}((M \otimes_R S)_{\mathfrak{b}}).$$

Replacing R by $R_{\mathfrak{a}}$, S by $S_{\mathfrak{b}}$, and M by $M_{\mathfrak{a}}$, we may assume that R, S are local and artinian and that f is a local map of local rings, in which case we must show that

$$\operatorname{length}_S(M \otimes_R S) = \operatorname{length}_R(M) \operatorname{length}_S(R/\mathfrak{a} \otimes_R S),$$

which is true as S is flat over R and M has a finite composition series whose factors are all isomorphic to $R/\mathfrak{a}$. $\square$

3. Types.

In this section, unless otherwise stated all representations will be over a fixed algebraic closure $\overline{E}$ of E. We say that a representation of W_F or I_F on a finite-dimensional $\overline{E}$-vector space V is *smooth* if it is continuous for the discrete topology on V, and *continuous* if it is continuous for the l-adic topology on V.

3.1. **Inertial types.** A Weil–Deligne representation of the Weil group W_F is a pair (r, N) where

- $r : W_F \to GL(V)$ is a smooth representation on a finite-dimensional vector space V;
- $N \in \operatorname{End}(V)$ satisfies

$$r(g)Nr(g)^{-1} = \|g\|N$$

  where $\| \cdot \| : W_F \twoheadrightarrow W_F/I_F \twoheadrightarrow q^{\mathbb{Z}}$ takes an arithmetic Frobenius element to q.

If $\rho : W_F \to GL(V)$ is a continuous representation of W_F on a finite-dimensional vector space V, then there is an associated Weil–Deligne representation (see for example [Tat79]) that we denote $\operatorname{WD}(\rho)$.

If $\rho : W_F \to GL(V)$ is a smooth irreducible representation of W_F on a finite dimensional vector space V and $k \geq 1$ is an integer, then define a Weil–Deligne representation $\operatorname{Sp}(\rho, k)$ by

$$\operatorname{Sp}(\rho, k) = (V \oplus V(1) \oplus \ldots \oplus V(k-1), N)$$

where for $0 \leq i \leq k - 2$, $N : V(i) \xrightarrow{\sim} V(i+1)$ is the isomorphism of vector spaces induced by some choice of basis for $\overline{E}(1)$, and $N(V(k-1)) = 0$.

Every Frobenius-semisimple Weil–Deligne representation (r, N) is isomorphic to one of the form

$$\bigoplus_{i=1}^{j} \mathrm{Sp}(\rho_i, k_i)$$

for smooth irreducible representations $\rho_i : W_F \to GL(V_i)$ and integers $j \geq 0$ and $k_i \geq 1$ for $i = 1, \ldots, j$. Up to obvious reorderings, the integers j and k_i are unique, and the representations ρ_i are unique up to isomorphism.

Definition 3.1. An **inertial type** is an isomorphism class of finite dimensional continuous representations τ of I_F such that there exists a continuous representation ρ of W_F with $\rho|_{I_F} \cong \tau$.

3.2. The classification of (Frobenius-semisimple) Weil–Deligne representations yields a classification of inertial types, which we now describe.

Definition 3.2. The set $\mathcal{I}_0$ of **basic** inertial types is the set of inertial types τ_0 that extend to a continuous *irreducible* representation of G_F.

Note that the τ_0 do not need to be irreducible representations of I_F.

Lemma 3.3. *Suppose that t, t' are positive integers, $\rho_1, \ldots, \rho_t, \rho'_1, \ldots, \rho'_{t'}$ are irreducible representations of W_F, and $k_1, \ldots, k_t, k'_1, \ldots, k'_t$ are positive integers. Then the representations of W_F associated to*

$$\bigoplus_{i=1}^{t} \mathrm{Sp}(\rho_i, k_i)$$

and

$$\bigoplus_{i=1}^{t'} \mathrm{Sp}(\rho'_i, k'_i)$$

have isomorphic restrictions to I_F if and only if $t = t'$ and there is an ordering $j_1, \ldots, j_t$ of $1, \ldots, t$ such that $k_i = k'_{j_i}$ and $\rho_i|_{I_F} \cong \rho'_{j_i}|_{I_F}$ for each $1 \leq i \leq t$.

Proof. The "if" direction is clear. We show the "only if" direction. If ρ is a continuous representation of W_F with $\mathrm{WD}(\rho) = (r, N)$, then $r|_{I_F}$ and the $r|_{I_F}$-equivariant endomorphism N are determined up to isomorphism by $\rho|_{I_F}$ (this follows from the construction of $\mathrm{WD}(\rho)$, see [Tat79] Corollary 4.2.2). So we may assume that $\rho = r$, so that all the k_i are zero. Now use the fact (proved by an exercise in Clifford theory) that, if ρ is an irreducible representation of W_F, then

$$\rho|_{I_F} \cong \mu_1 \oplus \ldots \oplus \mu_s$$

for some integer s and pairwise non-isomorphic irreducible representations μ_i of I_F which are in a single orbit for the action of G_F/I_F on irreducible representations of I_F; the representation μ_1 determines $\rho|_{I_F}$. Therefore, if

$$\bigoplus_{i=1}^{t} \rho_i|_{I_F} \cong \bigoplus_{i=1}^{t'} \rho'_i|_{I_F}$$

then $\rho_1|_{I_F}$ has an irreducible component in common with some $\rho'_{j_1}|_{I_F}$, and so $\rho_1|_{I_F} \cong \rho'_{j_1}|_{I_F}$. The lemma follows by induction. $\qquad \square$

Let Part be the set of integer sequences $P = (P(1), P(2), \ldots)$ which are decreasing and eventually zero. We regard $P \in$ Part as a partition of the integer $\deg(P) = \sum_{i=1}^{\infty} P(i)$. For each $\tau_0 \in \mathcal{I}_0$, choose an extension ρ_{τ_0} of τ_0 to W_F.

Definition 3.4. Let $\mathcal{I}$ be the set of functions $\mathcal{P} : \mathcal{I}_0 \to$ Part with finite support. For $\mathcal{P} \in \mathcal{I}$ we can form the Weil–Deligne representation

$$\bigoplus_{\tau_0 \in \mathcal{I}_0} \bigoplus_{i=0}^{\infty} \mathrm{Sp}(\rho_{\tau_0}, \mathcal{P}(\tau_0)(i)).$$

We define $\tau_{\mathcal{P}}$ to be the restriction to I_F of the associated representation of W_F; it is an inertial type.

By Lemma 3.3, the isomorphism class of $\tau_{\mathcal{P}}$ is independent of the choices of the ρ_{τ_0}, and the map $\mathcal{P} \mapsto \tau_{\mathcal{P}}$ is a bijection between $\mathcal{I}$ and the set of inertial types. To $\mathcal{P} \in \mathcal{I}$ we associate the function $\mathrm{scs}(\mathcal{P}) : \mathcal{I}_0 \to \mathbb{Z}_{\geq 0}$ given by $\mathrm{scs}(\mathcal{P})(\tau_0) = \deg \mathcal{P}(\tau_0)$. If $\tau = \tau_{\mathcal{P}}$ we write $\mathrm{scs}(\tau) = \mathrm{scs}(\mathcal{P})$.

Let $\succeq$ be the dominance order on Part; that is, the partial order defined by $P_1 \succeq P_2$ if and only if $\deg P_1 = \deg P_2$ and, for all $k \geq 1$,

$$\sum_{i=1}^{k} P_1(i) \geq \sum_{i=1}^{k} P_2(i).$$

Then $\succeq$ induces a partial order on $\mathcal{I}$ for which $\mathcal{P} \succeq \mathcal{P}'$ if and only if $\mathcal{P}(\tau_0) \succeq \mathcal{P}'(\tau_0)$ for all $\tau_0 \in \mathcal{I}_0$; we also sometimes regard $\succeq$ as a partial order on the set of inertial types.

3.3. **Fixed type deformation rings.** Let $\bar{\rho} : G_F \to GL_n(\mathbb{F})$ be a continuous representation and let τ be an inertial type. Suppose moreover that τ is defined over E, and that E contains all the $(q^{n^!} - 1)$th roots of unity. We say that a morphism $x : \operatorname{Spec} \overline{E} \to \operatorname{Spec} R^{\square}(\bar{\rho})$ has type τ if the corresponding Galois representation $\rho_x : G_F \to GL_n(\overline{E})$ does. Since τ is defined over E this only depends on the image of x.

Definition 3.5. If τ and $\bar{\rho}$ are as above, then $R^{\square}(\bar{\rho}, \tau)$ is the reduced quotient of $R^{\square}(\bar{\rho})$ such that $\operatorname{Spec} R^{\square}(\bar{\rho}, \tau)$ is the Zariski closure in $\operatorname{Spec} R^{\square}(\bar{\rho})$ of the $\overline{E}$-points of type τ.

If x is an $\overline{E}$-point of $\operatorname{Spec} R^{\square}(\bar{\rho}, \tau)$, say that x is **non-degenerate** if the associated Galois representation ρ_x satisfies $\mathrm{WD}(\rho_x) = \mathrm{rec}_l(\pi)$ for an irreducible admissible representation π of $GL_n(F)$ that is *generic* (see below for the defininitions of rec_l and generic).

Proposition 3.6. *For each inertial type τ defined over E:*

(1) $\operatorname{Spec} R^{\square}(\bar{\rho}, \tau)$ *is a union of irreducible components of* $\operatorname{Spec} R^{\square}(\bar{\rho})$;
(2) *if x is a non-degenerate $\overline{E}$-point of $\operatorname{Spec} R^{\square}(\bar{\rho})$, then x lies on a unique irreducible component of $\operatorname{Spec} R^{\square}(\bar{\rho})$ and $R^{\square}(\bar{\rho})[1/l]$ is formally smooth at x;*
(3) *the non-degenerate $\overline{E}$-points are Zariski dense in* $\operatorname{Spec} R^{\square}(\bar{\rho})$;
(4) *if x is a non-degenerate $\overline{E}$-point of $\operatorname{Spec} R^{\square}(\bar{\rho}, \tau)$, then ρ_x has type τ.*

Proof. Parts 2–4 follow from [BLGGT14] Lemmas 1.3.2 and 1.3.4. To show the first part we use Proposition 2.6. Firstly, note that under the isomorphism of Lemma 2.3 we have that

$$R^\square(\bar{\rho}, \tau) \cong \left(\widehat{\bigotimes}_{[\theta]} R^\square(\bar{\rho}_\theta, \tau_\theta) \right) [[X_1, \ldots, X_{n^2 - \sum n_\theta^2}]]$$

for some tamely ramified inertial types τ_θ, and so (by [BLGHT11] lemma 3.3) it suffices to prove the claim in the case that $\bar{\rho}$ is tamely ramified.

So suppose that $\bar{\rho}$ is tamely ramified. From our choices of topological generators σ, ϕ of G_F/I_F we have, as in the proof of Theorem 2.5, that $R^\square(\bar{\rho})$ is the completed local ring of $\mathcal{M}(n, q)_\mathcal{O}$ at the closed point of the special fibre corresponding to $\bar{\rho}$; in particular we have a flat morphism $\operatorname{Spec} R^\square(\bar{\rho}) \hookrightarrow \mathcal{M}(n, q)_\mathcal{O}$. Let C be the conjugacy class in $GL_n(\overline{E})$ of $\tau(\sigma)$. Then in Proposition 2.6 we defined the irreducible component $\mathcal{M}(n, q, \Sigma \sim C)_E$ of $\mathcal{M}(n, q)_E$; let $\mathcal{M}(n, q, \Sigma \sim C)_\mathcal{O}$ be its closure in $\mathcal{M}(n, q)_\mathcal{O}$. Then

$$\operatorname{Spec} R^\square(\bar{\rho}, \tau) = \operatorname{Spec} R^\square(\bar{\rho}) \cap \mathcal{M}(m, q, \Sigma \sim C)_\mathcal{O}$$

is a union of irreducible components of $\operatorname{Spec} R^\square(\bar{\rho})$ by the going down theorem. $\square$

3.4. **K-types.** Recall the local Langlands correspondence rec_F of [HT01] Theorem A, which is defined over the complex numbers. If π is an irreducible admissible $\overline{\mathbb{Q}}_l$-representation of $GL_n(F)$ and $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ is our choice of isomorphism, let

$$\operatorname{rec}_l(\pi) = \iota^{-1} \circ \operatorname{rec}_F(\iota \circ (\pi \otimes |\det|^{\frac{1-n}{2}})).$$

Then $\operatorname{rec}_l(\pi)$ is an n-dimensional Frobenius-semisimple Weil–Deligne representation of W_F over $\overline{\mathbb{Q}}_l$ and is independent of the choice of ι (see [Hen01] §7.4).

If $\mathcal{S} : \mathcal{I}_0 \to \mathbb{Z}_{\geq 0}$ is a function with finite support such that

$$\sum_{\tau_0 \in \mathcal{I}_0} \dim \tau_0 \mathcal{S}(\tau_0) = n,$$

then we can consider the full subcategory $\Omega_\mathcal{S}$ of $\operatorname{Rep}_{\overline{E}}(GL_n(F))$ all of whose irreducible subquotients π satisfy

$$\operatorname{scs}(\operatorname{rec}_l(\pi)|_{I_F}) = \mathcal{S}.$$

The category $\operatorname{Rep}_{\overline{E}}(GL_n(F))$ is then the direct product of the $\Omega_\mathcal{S}$; these are the **Bernstein components** of $\operatorname{Rep}_{\overline{E}}(GL_n(F))$. See, for example, [BK98] §1.

It is one of the main results of the theory of Bushnell and Kutzko developed in [BK93] and [BK99] that, for each Bernstein component Ω of

$$\operatorname{Rep}_{\overline{E}}(GL_n(F)),$$

there is a compact open subgroup $J \subset GL_n(F)$ and a representation λ of J with the following property: if $\pi \in \operatorname{Rep}_{\overline{E}}(GL_n(F))$ is generated by its λ-isotypic vectors, then π is in Ω. We call (J, λ) a 'type' for the Bernstein component Ω. If $K \supset J$ is a maximal compact subgroup of $GL_n(F)$ and Ω is supercuspidal, then $\operatorname{Ind}_J^K \lambda$ is irreducible and is a *K-type* for Ω.

In [SZ99], Schneider and Zink refine this by providing K-types for a certain 'stratification' of $\operatorname{Rep}_{\overline{E}}(GL_n(F))$. We use their results in the following Galois–theoretic form (c.f. [BC09] Proposition 6.3.3):

Theorem 3.7. *Let τ be an inertial type of dimension n. Then there is a smooth irreducible $\overline{E}$-representation $\sigma(\tau)$ of $GL_n(\mathcal{O}_F)$ such that, for each irreducible admissible $\overline{E}$-representation π of $GL_n(F)$, we have:*

(1) *if $\pi|_{GL_n(\mathcal{O}_F)}$ contains $\sigma(\tau)$, then $\mathrm{rec}_l(\pi)|_{I_F} \preceq \tau$;*
(2) *if $\mathrm{rec}_l(\pi)|_{I_F} \cong \tau$, then $\pi|_{GL_n(\mathcal{O}_F)}$ contains $\sigma(\tau)$ with multiplicity one;*
(3) *if $\mathrm{rec}_l(\pi)|_{I_F} \preceq \tau$ and π is generic, then $\pi|_{GL_n(\mathcal{O}_F)}$ contains $\sigma(\tau)$.*

Proof. This is [BC09] Proposition 6.3.3, except that we have replaced the hypothesis 'tempered' with 'generic'. That we can do this follows from the proof of [SZ99] Proposition 5.10 — the only property of tempered representations that is used is that they occur as the *irreducible* parabolic induction of a discrete series representation, and this continues to hold for generic representations. See also Corollary 6.21 below. $\square$

Example 3.8. Let $\mathcal{P}_0, \mathcal{P}_1 \in \mathcal{I}$ be the maps that take the trivial representation to (respectively) $(1, 1, 0, 0, \ldots)$ and $(2, 0, 0, \ldots)$, and everything else to zero. Let τ_0 and τ_1 be the corresponding inertial types; they are respectively the trivial two-dimensional representation and the non-trivial unipotent two-dimensional representation of I_F. We have $\mathcal{P}_0 \prec \mathcal{P}_1$ and they are not comparable to any other elements of $\mathcal{I}$.

The representation $\sigma(\tau_0)$ is the trivial representation of $GL_2(\mathcal{O}_F)$, while $\sigma(\tau_1)$ is inflated from the Steinberg representation of $GL_2(k_F)$.

Then π contains $\sigma(\tau_0)$ if and only if π is unramified, and so if and only if $\mathrm{rec}_l(\pi)|_{I_F} = \tau_0$. On the other hand, π containing $\sigma(\tau_1)$ implies that $\mathrm{rec}_l(\pi)|_{I_F}$ is unipotent — that is to say, that $\mathrm{rec}_l(\pi)|_{I_F} \preceq \tau_1$ — but the converse is false for π an unramified character (these are non-generic).

Remark 3.9. In fact we give a slightly different construction of the $\sigma(\tau)$, more amenable to reduction modulo l — this is the purpose of section 6, and see Corollary 6.21 for a proof that the representations we construct have the desired properties (modulo the translation into Galois theoretic language, which is straightforward and exactly as in [BC09]). It seems plausible that the two constructions coincide, but we do not need this and have not checked it.

3.5. **The Bernstein–Zelevinsky classification.** It will be useful to recall a little notation to do with the Bernstein–Zelevinsky classification of irreducible admissible representations of $GL_n(F)$; we follow [Rod82]. For definiteness, fix a choice of square root of q in $\overline{E}$. Then if $P \subset GL_n(F)$ is a standard parabolic subgroup with Levi factor $M = \prod_{i=1}^{k} M_i$ and unipotent radical U, and if ρ_i are smooth representations of M_i, we can regard $\otimes_i \rho_i$ as a representation of P by allowing U to act trivially and then form the normalised parabolic induction of $\otimes_i \rho_i$ from P to $GL_n(F)$; call this representation

$$\rho_1 \times \ldots \times \rho_k.$$

If π is an irreducible supercuspidal representation of $GL_m(F)$ and $k \geq 1$ is an integer, let

$$\Delta(\pi, k) = \{\pi, \pi \otimes |\det|, \ldots, \pi \otimes |\det|^{k-1}\}.$$

A set of this form is called a **segment**. Two segments Δ_1 and Δ_2 are called **linked** if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$ and $\Delta_1 \cup \Delta_2$ is a segment, and we say that $\Delta(\pi, k)$ **precedes**

$\Delta(\pi', k')$ if they are linked and $\pi' = \pi \otimes |\det|^s$ for some $s \geq 1$. If $\Delta = \Delta(\pi, k)$ is a segment, let $L(\Delta)$ be the unique irreducible quotient of

$$\pi \times (\pi \otimes |\det|) \times \ldots \times (\pi \otimes |\det|^{k-1});$$

it is an irreducible admissible representation of $GL_{km}(F)$. If $\Delta_1, \ldots, \Delta_t$ are segments then we may reorder them so that, for $i < j$, Δ_i does not precede Δ_j. Then

$$L(\Delta_1) \times \ldots \times L(\Delta_t)$$

is a representation of $GL_n(F)$ for suitable n, with a unique irreducible quotient $L(\Delta_1, \ldots, \Delta_t)$, which is independent of the ordering chosen (so long as the 'precedence' condition is satisfied). Every irreducible admissible representation of $GL_n(F)$ is of this form, uniquely up to reordering the Δ_i. The representation

$$L(\Delta_1) \times \ldots \times L(\Delta_t)$$

is irreducible if and only if no two of the Δ_i are linked. In this case $L(\Delta_1, \ldots, \Delta_t) = L(\Delta_1) \times \ldots \times L(\Delta_t)$ is **generic**, and moreover every irreducible generic representation is of this form (see [Zel80] Theorem 9.7).

The compatibility with the above classification of Frobenius-semisimple Weil–Deligne representations is as follows. If $d_1, \ldots, d_t$ are positive integers with $\sum d_i = n$, $\pi_1, \ldots, \pi_t$ are supercuspidal representations of $GL_{d_i}(F)$, and $k_1, \ldots, k_t$ are positive integers, then for

$$\Delta_i = \Delta(\pi_i \otimes |\det|^{\frac{1-d_i}{2}}, k_i)$$

we have:

$$(3) \qquad \bigoplus_{i=1}^{t} \mathrm{Sp}(\mathrm{rec}_l(\pi_i), k_i) = \mathrm{rec}_l(|\det|^{\frac{n-1}{2}} \otimes L(\Delta_1, \ldots, \Delta_t)).$$

The next two paragraphs are only required in section 6. A **supercuspidal pair** is a pair (M, π) where M is a Levi subgroup of some $GL_n(F)$ and π is a supercuspidal representation of M. We say that supercuspidal pairs (M, π) and (M', π') are **inertially equivalent** if there is an element $g \in G$ and an unramified character α of M' such that $M' = gMg^{-1}$ and $\pi' = \alpha \otimes \pi^g$. We write $[M, \pi]$ for the inertial equivalence class of (M, π). If Ω is a Bernstein component of $\mathrm{Rep}_{\overline{E}}(GL_n(F))$, then there is a unique inertial equivalence class of supercuspidal pair $[M, \pi]$ such that every irreducible object of Ω is a subquotient of a representation parabolically induced from a supercuspidal pair (M, π) in that inertial equivalence class (for some choice of parabolic subgroup).

The **essentially discrete series** representations[4] of $GL_n(F)$ are precisely those of the form $L(\Delta)$ for some segment Δ. Define a **discrete pair** to be a pair (M, π) where M is a Levi subgroup of some $GL_n(F)$ and π is an essentially discrete series representation of M; say that discrete pairs (M, π) and (M', π') are inertially equivalent if there is an element $g \in G$ and an unramified character α of M' such that $M' = gMg^{-1}$ and $\pi' = \alpha \otimes \pi^g$, and write $[M, \pi]$ for the inertial equivalence class of (M, π). If π is supercuspidal this agrees with the notion of inertial equivalence for supercuspidal pairs. If $\mathcal{P} \in \mathcal{I}$ then we can associate an inertial equivalence

---

[4]That is, the unramified twists of discrete series representations

class $[M, \pi]$ of discrete pairs to $\mathcal{P}$ as follows: for every $\tau_0 \in \mathcal{I}_0$ pick a supercuspidal representation π_{τ_0} of $GL_{\dim(\tau_0)}(F)$ with $\mathrm{rec}_l(\pi_{\tau_0})|_{I_F} \cong \tau_0$. Then

$$\left[\prod_{\tau_0 \in \mathcal{I}_0, i \in \mathbb{N}} GL_{\mathcal{P}(\tau_0)(i)\dim(\tau_0)}(F), \quad \bigotimes_{\tau_0 \in \mathcal{I}_0, i \in \mathbb{N}} L(\Delta(\pi_{\tau_0}, \mathcal{P}(\tau_0)(i))) \right]$$

is the required class of discrete pairs. If $(M, \pi) = (\prod_{i=1}^r M_i, \bigotimes_{i=1}^r L(\Delta_i))$ is a discrete pair, then we can define $L(M, \pi)$ to be $L(\Delta_1, \ldots, \Delta_r)$. From equation (3) we see that, if $\mathcal{P} \in \mathcal{I}$ has degree n, and $[M, \pi]$ is the associated inertial equivalence class of discrete pair, then the irreducible admissible representations π of $GL_n(F)$ such that $\mathrm{rec}_l(\pi)$ has type $\tau_\mathcal{P}$ are precisely the $L(M, \pi)$ for (M, π) in the inertial equivalence class $[M, \pi]$.

4. The Breuil–Mézard conjecture.

4.1. Reduction maps.
Let $\bar{\rho} : G_F \to GL_n(\mathbb{F})$ be a continuous representation, and suppose that E is large enough that, for every inertial type τ that is the type of some lift of $\bar{\rho}$, both τ and $\sigma(\tau)$ are defined over E. We have defined the framed deformation ring $R^\square(\bar{\rho})$, which is flat and equidimensional of relative dimension n^2 over $\mathcal{O}$. Thus we have the free abelian groups $\mathcal{Z}(R^\square(\bar{\rho}))$ on the irreducible components of $R^\square(\bar{\rho})$, $\mathcal{Z}(\overline{R}^\square(\bar{\rho}))$ on the irreducible components of $\overline{R}^\square(\bar{\rho}) = R^\square(\bar{\rho}) \otimes \mathbb{F}$, and a reduction map

$$\mathrm{red} : \mathcal{Z}(R^\square(\bar{\rho})) \to \mathcal{Z}(\overline{R}^\square(\bar{\rho})).$$

Let $R_E(GL_n(\mathcal{O}_F))$ be the Grothendieck group of finite-dimensional smooth representations of $GL_n(\mathcal{O}_F)$ over E, and let $R_{\mathbb{F}}(GL_n(\mathcal{O}_F))$ be the Grothendieck group of finite-dimensional smooth representations of $GL_n(\mathcal{O}_F)$ over $\mathbb{F}$. Then the operation of choosing a $GL_n(\mathcal{O}_F)$-invariant lattice and reducing modulo λ defines a group homomorphism:

$$\mathrm{red} : R_E(GL_n(\mathcal{O}_F)) \longrightarrow R_{\mathbb{F}}(GL_n(\mathcal{O}_F)).$$

4.2. Cycle map.

Lemma 4.1. *If π and π' are generic irreducible admissible representations of $GL_n(F)$ such that $\mathrm{rec}_l(\pi)|_{I_F} \cong \mathrm{rec}_l(\pi')|_{I_F}$, then*

$$\pi|_{GL_n(\mathcal{O}_F)} \cong \pi'|_{GL_n(\mathcal{O}_F)}.$$

Proof. Let $\mathcal{P} \in \mathcal{I}$ be such that $\mathrm{rec}_l(\pi)|_{I_F} \cong \mathrm{rec}_l(\pi')|_{I_F} \cong \tau_\mathcal{P}$ and let $\tau_1, \ldots, \tau_r$ be the elements of $\mathcal{I}_0$ with $\deg \mathcal{P}(\tau_i) = d_i \neq 0$. Pick supercuspidal representations π_i of $GL_{\dim \tau_i}(F)$ such that $\mathrm{rec}_l(\pi_i)|_{I_F} \cong \tau_i$ and let $\Delta_{i,j}$ be the segment $\Delta(\pi_i, \mathcal{P}(\tau_i)(j))$ for each j such that $\mathcal{P}(\tau_i)(j) \neq 0$. Then every generic irreducible admissible representation π of $GL_n(F)$ such that $\mathrm{rec}_l(\pi)|_{I_F} \cong \tau$ is of the form

$$(\alpha_{1,1} \circ \det)L(\Delta_{1,1}) \times \ldots \times (\alpha_{i,j} \circ \det)L(\Delta_{i,j}) \times \ldots$$

for unramified characters $\alpha_{i,j}$ of $F^\times$. The lemma follows from the following consequence of the Iwasawa decomposition: for any parabolic subgroup $P \subset GL_n(F)$ and representation ρ of P,

$$(\mathrm{Ind}_P^G \rho)|_{GL_n(\mathcal{O}_F)} = \mathrm{Ind}_{P \cap GL_n(\mathcal{O}_F)}^{GL_n(\mathcal{O}_F)}(\rho|_{P \cap GL_n(\mathcal{O}_F)}). \qquad \square$$

Definition 4.2. If θ is a finite-length representation of $GL_n(\mathcal{O}_F)$ and τ' is an inertial type, then $m(\theta, \tau')$ is defined to be the non-negative integer

$$\dim \operatorname{Hom}_{\overline{E}[GL_n(\mathcal{O}_F)]}(\theta, \pi|_{GL_n(\mathcal{O}_F)})$$

for any generic irreducible admissible representation π of $GL_n(F)$ such that

$$\operatorname{rec}_l(\pi)|_{I_F} \cong \tau'.$$

Proposition 4.3. *Suppose that* $\mathcal{P}, \mathcal{P}' \in \mathcal{I}$. *Then*

$$m(\sigma(\tau_{\mathcal{P}}), \tau_{\mathcal{P}'}) = \prod_{\tau_0 \in \mathcal{I}_0} m(\mathcal{P}(\tau_0), \mathcal{P}'(\tau_0))$$

where $m(\mathcal{P}(\tau_0), \mathcal{P}'(\tau_0))$ *is the Kostka number (Definition 6.2) for the pair of partitions* $\mathcal{P}(\tau_0), \mathcal{P}'(\tau_0)$ *(and is in particular zero if* $\deg \mathcal{P}(\tau_0) \neq \deg \mathcal{P}'(\tau_0)$ *for some* τ_0*).*

Proof. This is proved as Corollary 6.22 below. $\qquad\qquad\qquad\qquad\qquad\square$

Definition 4.4. Define a homomorphism

$$\operatorname{cyc} : R_E(GL_n(\mathcal{O}_F)) \longrightarrow \mathcal{Z}(R^\square(\overline{\rho}))$$

given (on irreducible E-representations σ of $GL_n(\mathcal{O}_F)$) by:

$$\operatorname{cyc}(\sigma) = \sum_{\mathcal{P} \in \mathcal{I}} m(\sigma^\vee, \tau_{\mathcal{P}}) Z(R^\square(\overline{\rho}, \tau_{\mathcal{P}})).$$

This sum makes sense since $m(\sigma^\vee, \tau_{\mathcal{P}})$ is non-zero for only finitely many $\tau_{\mathcal{P}}$.

Conjecture 4.5. *There exists a unique homomorphism*

$$\overline{\operatorname{cyc}} : R_{\mathbb{F}}(GL_n(\mathcal{O}_F)) \longrightarrow \mathcal{Z}(\overline{R}^\square(\overline{\rho}))$$

making the following diagram commute:

$$
\begin{array}{ccc}
R_E(GL_n(\mathcal{O}_F)) & \xrightarrow{\ \operatorname{cyc}\ } & \mathcal{Z}(R^\square(\overline{\rho})) \\
{\scriptstyle \operatorname{red}}\big\downarrow & & {\scriptstyle \operatorname{red}}\big\downarrow \\
R_{\mathbb{F}}(GL_n(\mathcal{O}_F)) & \xrightarrow{\ \overline{\operatorname{cyc}}\ } & \mathcal{Z}(\overline{R}^\square(\overline{\rho})).
\end{array}
$$

(4)

Certainly there is at most one map $\overline{\operatorname{cyc}}$ making diagram ((4) commute. This is because the map $\operatorname{red} : R_E(GL_n(\mathcal{O}_F)) \to R_{\mathbb{F}}(GL_n(\mathcal{O}_F))$ is surjective, which follows from the corresponding fact for finite groups (see [Ser77] Theorem 33).

The main result of this chapter is:

Theorem 4.6. *If* $l > 2$ *then Conjecture 4.5 is true.*

Proof. Let $\operatorname{Rep}_{\mathcal{O}}^{fg}(GL_n(\mathcal{O}_F))$ be the category of finitely generated $\mathcal{O}$-modules with a smooth representation of $GL_n(\mathcal{O}_F)$. In the next section, we will show (using the Taylor–Wiles–Kisin patching method) that there are positive integers c and d, a complete noetherian local domain A over $\mathcal{O}$, and an exact functor H_∞ from $\operatorname{Rep}_{\mathcal{O}}^{fg}(GL_n(\mathcal{O}_F)^{\times d})$ to the category of finitely generated modules over

$$R^\square(\overline{\rho})^{\otimes d} \hat{\otimes} A$$

with the following properties:

- for all $\sigma \in \operatorname{Rep}_{\mathcal{O}}^{fg}(GL_n(\mathcal{O}_F)^{\times d})$,

$$H_\infty(\sigma \otimes_{\mathcal{O}} \mathbb{F}) = H_\infty(\sigma) \otimes_{\mathcal{O}} \mathbb{F};$$

- if $\sigma \in \mathrm{Rep}_{\mathcal{O}}^{fg}$ is λ-torsion free, then so is $H_{\infty}(\sigma)$;
- if $\sigma = \bigotimes_{i=1}^{d} \sigma_i \in \mathrm{Rep}_{\mathcal{O}}^{fg}(GL_n(\mathcal{O}_F)^{\times d})$ is finite free as an $\mathcal{O}$-module, then

$$Z(H_{\infty}(\sigma)) = c \cdot \bigotimes_{i=1}^{d} \left(\sum_{\mathcal{P} \in \mathcal{I}} m(\sigma_i^{\vee}, \tau_{\mathcal{P}}) Z(R^{\square}(\overline{\rho}, \tau_{\mathcal{P}})) \right)$$
$$= c \cdot \mathrm{cyc}^{\otimes d}(\sigma)$$

where we identify $\mathcal{Z}(R^{\square}(\overline{\rho})^{\otimes d} \hat{\otimes} A))$ with $\bigotimes_{i=1}^{d} \mathcal{Z}(R^{\square}(\overline{\rho}))$, using that A is a domain.

Now, $Z(\cdot)$ is additive on short exact sequences (see [EG14] Lemma 2.2.7) and, using Lemma 2.8, we find that the following diagram commutes (the horizontal maps are well defined since $H_{\infty}(\cdot)$ is exact):

$$
\begin{array}{ccc}
R_E(GL_n(\mathcal{O}_F))^{\otimes d} & \xrightarrow{Z(H_{\infty}(\cdot))} & \mathcal{Z}(R^{\square}(\overline{\rho}))^{\otimes d} \\
{\scriptstyle \mathrm{red}^{\otimes d}} \downarrow & & {\scriptstyle \mathrm{red}^{\otimes d}} \downarrow \\
R_{\mathbb{F}}(GL_n(\mathcal{O}_F))^{\otimes d} & \xrightarrow{Z(H_{\infty}(\cdot))} & \mathcal{Z}(\overline{R}^{\square}(\overline{\rho}))^{\otimes d}.
\end{array}
$$

Moreover, the topmost map is just $c \cdot \mathrm{cyc}^{\otimes d}$ by the second listed property of H_{∞}. We deduce that $\ker(\mathrm{red}) \subset \ker(\mathrm{red} \circ \mathrm{cyc})$; if not, then we may pick $\alpha \in \ker(\mathrm{red})$ with $\beta = \mathrm{red}(\mathrm{cyc}(\alpha)) \neq 0$. But then

$$c \cdot \mathrm{red}^{\otimes d}(\mathrm{cyc}^{\otimes d}(\alpha \otimes \ldots \otimes \alpha)) = c(\beta \otimes \ldots \otimes \beta) \neq 0$$

and also

$$c \cdot \mathrm{red}^{\otimes d}(\mathrm{cyc}^{\otimes d}(\alpha \otimes \ldots \otimes \alpha)) = Z(H_{\infty}(\mathrm{red}(\alpha) \otimes \ldots \otimes \mathrm{red}(\alpha))) = 0,$$

a contradiction. From $\ker(\mathrm{red}) \subset \ker(\mathrm{red} \circ \mathrm{cyc})$ we immediately obtain the existence of the map $\overline{\mathrm{cyc}}$. $\qquad \square$

Remark 4.7. Let $\mathcal{T}$ be the subgroup of $R_E(GL_n(\mathcal{O}_F))$ generated by the $\sigma(\tau)$ for inertial types τ, and let $\overline{\mathcal{T}}$ be the subgroup of $R_{\mathbb{F}}(GL_n(\mathcal{O}_F))$ generated by those irreducible representations appearing as a constituent of some $\mathrm{red}(\sigma(\tau))$. Then $\mathrm{red} : \mathcal{T} \to \overline{\mathcal{T}}$ is surjective, by Theorem 6.23 below. It follows that the version of Theorem 4.6 in which $R_E(GL_n(\mathcal{O}_F))$ (resp. $R_{\mathbb{F}}(GL_n(\mathcal{O}_F))$) is replaced by $\mathcal{T}$ (resp. $\overline{\mathcal{T}}$) is also true — the only possible issue being the uniqueness of $\overline{\mathrm{cyc}}$. When $l > n$, $l \mid q - 1$, and $\overline{\rho}|_{\tilde{P}_F}$ is trivial, we prove a version of the theorem with a still further restricted choice of $\mathcal{T}$ in section 7 below, using local methods.

It is natural to ask to what extent Theorem 4.6 gives a 'formula' for the cycle $\mathrm{red}(Z(R^{\square}(\overline{\rho}, \tau)))$. This is answered by:

Proposition 4.8. *The image of* cyc *is the subgroup* $\mathcal{H}$ *of* $\mathcal{Z}(R^{\square}(\overline{\rho}))$ *spanned by the cycles* $Z(R^{\square}(\overline{\rho}, \tau))$ *for varying* τ. *Moreover, the restriction of* cyc *to* $\mathcal{T}$ *(see the previous remark) is a bijection onto* $\mathcal{H}$.

Proof. It is clear that the image of cyc is contained in $\mathcal{H}$. By Proposition 4.3 and basic properties of Kostka numbers, the matrix of multiplicities $m(\sigma(\tau), \tau')$ is upper triangular with '1's on the diagonal (for an appropriate ordering of the various τ). It follows that cyc restricted to $\mathcal{T}$ is a bijection onto $\mathcal{H}$, and thus that the image of cyc is all of $\mathcal{H}$. $\qquad \square$

Let cyc^{-1} be the inverse of $\mathrm{cyc} : \mathcal{T} \to \mathcal{H}$. For τ an inertial type, let $r(\tau) = \mathrm{cyc}^{-1}(Z(R^\square(\bar\rho, \tau)))$; then

$$r(\tau) = \sum_{\tau'} m^{-1}(\sigma(\tau), \tau')\sigma(\tau')$$

where $m^{-1}(\sigma(\tau), \tau')$ is the (τ, τ')-entry of the *inverse* of the matrix $(m(\sigma(\tau), \tau'))_{\tau,\tau'}$. Using Proposition 4.3, $m^{-1}(\sigma(\tau), \tau')$ can be written as a product of entries of the inverse to the matrix of Kostka numbers; moreover, it is zero unless τ and τ' have the same semisimplification. If τ is semisimple then $r(\tau) = \sigma(\tau)$, but in general it is only an element of the Grothendieck group. As an immediate consequence of Theorem 4.6 we have:

Corollary 4.9. *For each inertial type τ,*

$$\mathrm{red}(Z(R^\square(\bar\rho, \tau))) = \overline{\mathrm{cyc}}(r(\tau)).$$

This expresses $Z(R^\square(\bar\rho, \tau))$ in terms of the $\overline{\mathrm{cyc}}(\theta)$ for θ running over the irreducible $\overline{\mathbb{F}}$-representations of $GL_n(\mathcal{O}_F)$. We say nothing here about the determination of the $\overline{\mathrm{cyc}}(\theta)$.

Example 4.10. Let $n = 2$. As in Example 3.8, let τ_0 and τ_1 be respectively the trivial and non-trivial two-dimensional unipotent representations of I_F. Then $\sigma(\tau_0) = \mathbb{1}$ is the trivial representation and $\sigma(\tau_1) = \mathrm{St}$ is inflated from the Steinberg representation of $GL_2(k_F)$. Then $r(\tau_0) = \sigma(\tau_0) = \mathbb{1}$, while

$$r(\tau_1) = \mathrm{St} - \mathbb{1}.$$

Example 4.11. Let $n = 3$. Let τ_0, τ_1 and τ_2 be the three-dimensional unipotent representations of I_F for which the Weil–Deligne monodromy operator N has rank 0, 1 and 2, respectively. Then $\sigma(\tau_0)$, $\sigma(\tau_2)$ are the inflations to $GL_3(\mathcal{O}_F)$ of, respectively, the trivial representation and the Steinberg representation of $GL_3(k_F)$; $\sigma(\tau_1)$ is then inflated from the remaining irreducible unipotent representation of $GL_3(k_F)$. Then the representations $r(\tau_i)$ are as follows:

$$r(\tau_0) = \sigma(\tau_0)$$
$$r(\tau_1) = \sigma(\tau_1) - 2\sigma(\tau_0)$$
$$r(\tau_2) = \sigma(\tau_2) - \sigma(\tau_1) + \sigma(\tau_0).$$

Remark 4.12. If multiple components of $\mathrm{Spec}\, R^\square(\bar\rho)$ have the same type, then $\mathcal{H}$ will be a strict subgroup of $Z(R^\square(\bar\rho, \tau))$; this happens, for instance, if $n = 2$, $\bar\rho = \mathbb{1} \oplus \chi$ where χ is the cyclotomic character, and $q \equiv -1 \bmod l$ (see [Sho13] Proposition 5.6).

Remark 4.13. In the $l = p$ setting, [EG14] Conjecture 4.1.6 deals only with the potentially crystalline situation. To obtain the most general conjecture in the potentially semistable case, one would also have to use the map cyc.

5. Global proof.

For the entirety of this section, we assume that $l > 2$.

5.1. **Automorphic forms.** We define the spaces of automorphic forms on definite unitary groups that we will patch using the Taylor–Wiles–Kisin method. See also [CHT08], [EG14], [Ger10], [Tho12]. Our reason for reproducing this now standard material here is that we need to allow more general level at places $v \nmid l$ than is considered in those references; hopefully it will be clear that there is no essential difference.

5.1.1. Let L be an imaginary CM field with maximal totally real subfield L^+ satisfying the following hypotheses:

  (1) $[L^+ : \mathbb{Q}]$ is divisible by 4;
  (2) L/L^+ is unramified at all finite places;
  (3) every place $v \mid l$ of L^+ splits in L.

Let δ_{L/L^+} be the non-trivial character of G_L that is trivial G_{L^+}, and let c be the non-trivial element of $\mathrm{Gal}(L^+/L)$. Then as in [Tho12] section 6 (see also [CHT08] section 3.3), we may choose a group scheme G over $\mathcal{O}_{L^+}$ and an L^+-linear involution $*$ on $M_n(L)$ such that:

  - $(xy)^* = y^* x^*$ for all $x, y \in M_n(L)$ and $x^* = x^c$ for $x \in Z(M_n(L)) \cong L$;
  - for any L^+-algebra R,
  $$G(R) = \{g \in M_n(L) \otimes_{L^+} R : g^* g = 1\};$$
  - for every finite place v of L^+, $G \times_{L^+} L_v^+$ is quasi-split;
  - for every infinite place v of L^+, $G(L_v^+) \cong U_n(\mathbb{R})$, the compact unitary group;
  - there is a maximal order $A \subset M_n(L)$ with $A^* = A$ and $G(\mathcal{O}_{L^+}) = G(L^+) \cap A$;
  - for v a finite place of L^+ split as ww^c in L there is an isomorphism
  $$\iota_w : G(L_v^+) \to GL_n(L_w)$$
  such that $\iota_w(G(\mathcal{O}_{L_v^+})) = GL_n(\mathcal{O}_{L_w})$ and $\iota_{w^c}(x) = ({}^t\iota_w(x)^c)^{-1}$.

Let S be a set of finite places of L^+ split in L, and let S_l be the set of places of L^+ above l (we may or may not have $S \cap S_l = \emptyset$). Write $T = S \cup S_l$. Suppose that U is a subgroup of $G(\mathbb{A}_{L^+}^\infty)$, and write U_v for the image of the projection of U to $G(L_v^+)$. Call U **good** if it is compact and if:

  - for $v \in T$, $U_v \subset G(\mathcal{O}_{L_v^+})$;
  - for some $v \in S$, the only element of U_v of finite order is the identity element (U_v is 'sufficiently small' in the language of [CHT08]).

5.1.2. For $v \in S$, let M_v be an $\mathcal{O}$-module with an $\mathcal{O}$-linear action of $G(\mathcal{O}_{L_v^+})$ which is continuous for the discrete topology on M_v.

Suppose that E contains the images of all embeddings $L \hookrightarrow \overline{E}$. Let $I_l = \mathrm{Hom}(L^+, E)$, so that I_l surjects onto S_l with $\theta \in I_l$ mapping to a place $v(\theta)$ of L^+ above l. For each $v \in S_l$ pick a place $\tilde{v}$ of L above v, let $\tilde{S}_l = \{\tilde{v} : v \in S_l\}$ and let $\tilde{I}_l$ be the set of embeddings $L \hookrightarrow E$ inducing a place of $\tilde{S}_l$. Restriction to L^+ defines a bijection $\tilde{I}_l \xrightarrow{\sim} I_l$. Now let
$$\mathbb{Z}_+^n = \{(\lambda_1, \dots \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \dots \geq \lambda_n\}.$$

As in [Ger10], we associate to each $\lambda \in \mathbb{Z}_+^n$ a representation ξ_λ (defined over $\mathcal{O}$) of $GL_n/\mathcal{O}$, and let $M_\lambda = \xi_\lambda(\mathcal{O})$ and $V_\lambda = \xi_\lambda(E)$.

Now suppose that $\lambda = (\lambda_\theta)_\theta \in (\mathbb{Z}_+^n)^{\tilde{I}_l}$. Then we define

$$M_\lambda = \otimes_\theta M_{\lambda_\theta}$$

and regard this as a representation of $\prod_{v \in S_l} G(\mathcal{O}_{L_v^+})$ via the product of the composites of the maps

$$G(\mathcal{O}_{L_{v(\theta)}^+}) \xrightarrow{\iota_{\tilde{v}(\theta)}} GL_n(\mathcal{O}_{L_{\tilde{v}(\theta)}}) \xrightarrow{\theta} GL_n(\mathcal{O}) \xrightarrow{\xi_{\lambda_\theta}} GL(M_{\lambda_\theta}).$$

Finally let $M = \bigotimes_{v \in S} M_v$, a representation of $\prod_{v \in S} G(\mathcal{O}_{L_v^+})$ and hence (by projection) of any good subgroup U; we also consider the representation $M \otimes M_\lambda$ of $\prod_{v \in T} G(\mathcal{O}_{L_v^+})$ and hence of U.

Definition 5.1. Suppose that U is a good *open* subgroup. Then $S_\lambda(U, M)$ is the space of functions

$$f : G(L^+) \backslash G(\mathbb{A}_{L^+}^\infty) \to M \otimes M_\lambda$$

such that $f(gu) = u^{-1} f(g)$ for all $u \in U$.

If V is a good subgroup of $G(\mathbb{A}_{L^+})^\infty$ then define

$$S_\lambda(V, M) = \varinjlim S_\lambda(U, M)$$

where the limit runs over all good open subgroups U containing V.

If M is a finitely generated $\mathcal{O}$-module then $S_\lambda(U, M)$ is a finitely generated $\mathcal{O}$-module, because $G(L^+) \backslash G(\mathbb{A}_{L^+}^\infty)/U$ is a finite set.

Lemma 5.2. *Let U be a good open subgroup.*

(1) *The functor*

$$(M_v)_{v \in T} \longmapsto S_\lambda(U, \bigotimes_v M_v \otimes M_\lambda)$$

is exact.

(2) *If $V \subset U$ is a normal, good, open subgroup, then there are isomorphisms of $\mathcal{O}$-modules*

$$S_\lambda(V, M) \longrightarrow S_\lambda(U, M) \otimes_\mathcal{O} \mathcal{O}[U/V]$$

and

$$S_\lambda(V, M)_{U/V} \xrightarrow{\mathrm{tr}_{U/V}} S_\lambda(U, M)$$

where $S_\lambda(V, M)_{U/V}$ denotes the U/V-coinvariants in $S_\lambda(V, M)$.

Proof. This may be proved by the argument of [CHT08] Lemma 3.3.1, using that U is sufficiently small. □

5.1.3. *Hecke operators.* Now suppose that $U = U_S U^S$ where $U_S \subset \prod_{v \in S} G(\mathcal{O}_{L_v^+})$ and $U^S = \prod_{v \notin S} U_v$ where $U_v \subset G(L_v^+)$ for each $v \notin S$. Suppose also that $S \cap S_l = \emptyset$ and that for finite places $v \notin T$ of L^+ split in L we have $U_v = G(\mathcal{O}_{L_v^+})$. We define Hecke operators, following [Ger10], section 2.3.

Definition 5.3.   (1) Let $v \notin T$ be a place of L^+ splitting as ww^c in L. Then for $1 \leq j \leq n$ define the operator $T_w^{(j)}$ on $S_\lambda(U, M)$ as the double coset operator:

$$T_w^{(j)} = \left[U \iota_w^{-1} \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} U \right]$$

for some (any) choice of uniformiser ϖ_w of $\mathcal{O}_{L_w}$, where 1_j is the $j \times j$ identity matrix.

(2) For $v \in S_l$, w a place of L above v, and $\varpi_w \in \mathcal{O}_{L_w}$ a uniformiser, define:

$$T_{\lambda,\varpi_w}^{(j)} = \left((w_0\lambda) \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \right)^{-1} \left[U\iota_w^{-1} \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} U \right]$$

where $w_0\lambda$ is the conjugate of λ by the longest element w_0 of the Weyl group.

Let $\mathbb{T}^T$ be the polynomial ring over $\mathcal{O}$ generated by all the $T_w^{(j)}$ and $(T^{(n)})^{-1}$. Let $\mathbb{T}_\lambda^T(U, M)$ be image of $\mathbb{T}^T$ in $\mathrm{End}(S_\lambda(U, M))$, and let $\tilde{\mathbb{T}}_\lambda^T(U, M)$ be the $\mathcal{O}$-subalgebra of $\mathrm{End}(S_\lambda(U, M))$ generated by $\mathbb{T}_\lambda^T(U, M)$ and by all $T_{\lambda,\varpi_w}^{(j)}$ for $w \in S_l$.

Say that a maximal ideal $\mathfrak{m}$ of $\tilde{\mathbb{T}}_\lambda^{T_l}(U, M)$ is **ordinary** if each $T_{\lambda,\varpi_w}^{(j)}$ has non-zero image in $\tilde{\mathbb{T}}_\lambda^T(U, M)/\mathfrak{m}$. Say that a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\lambda^T(U, M)$ is **ordinary** if $\mathfrak{m} = \mathfrak{m}' \cap \mathbb{T}_\lambda^T(U, M)$ for an ordinary maximal ideal $\mathfrak{m}'$ of $\tilde{\mathbb{T}}_\lambda^T(U, M)$.

5.1.4. *Base change.* Keep the assumptions of the previous section, and suppose also that U_v is a hyperspecial maximal compact subgroup of $G(L_v^+)$ for each place v of L^+ inert in L and that M is a finite free $\mathcal{O}$-module. Suppose that $\mathcal{A}$ is the space of (complex-valued) automorphic forms on $G(\mathbb{A}_{L^+})$ and that $\pi = \bigotimes_v' \pi_v$ is an irreducible constituent of $\mathcal{A}$ with weight $\lambda_\infty \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(L^+, \mathbb{C})}$ such that (recalling the fixed isomorphism $\iota : \overline{E} \xrightarrow{\sim} \mathbb{C}$):

- for $\theta \in \mathrm{Hom}(L^+, \mathbb{C})$, $(\lambda_\infty)_\theta = \lambda_{\iota \circ \theta}$;
- for $v \notin S$ a place of L^+, $\pi_v^{U_v} \neq 0$;
- for $v \notin S$ a place of L^+ split as ww^c in L, $T_w^{(j)}$ acts as a scalar $\iota(a_w^{(j)})$ for some $a_w^{(j)} \in \overline{E}$.

Let $f_\pi : \mathbb{T}^T \to \overline{E}$ be the homomorphism taking $T_w^{(j)}$ to $a_w^{(j)}$.

Lemma 5.4. *Suppose that π, U, λ and M satisfy the above hypotheses. Then we have the formula:*

$$\dim\left(S_\lambda(U, M) \otimes_{\mathbb{T}^T, f_\pi} \overline{E} \right) = \dim \mathrm{Hom}_{U_S}\left((M \otimes_\mathcal{O} \overline{E})^\vee, \bigotimes_{v \in S} \pi_v \otimes_{\mathbb{C}, \iota^{-1}} \overline{E} \right).$$

Proof. (sketch) As in [CHT08] Proposition 3.3.2, we have:

$$S_\lambda(U, M) \otimes_{\mathcal{O}, \iota} \mathbb{C} \cong \mathrm{Hom}_{U_S \times G(L_\infty^+)}\left((M \otimes_{\mathcal{O}, \iota} \mathbb{C})^\vee \otimes V_\infty^\vee, \mathcal{A}^{U^S} \right)$$

where V_∞ is an algebraic representation of $G(L_\infty^+)$ constructed from λ_∞. It suffices to show that any other irreducible $\pi' \subset \mathcal{A}$ satisfying the above three conditions (for the same values of $a_w^{(j)}$) is actually equal to π. By [Lab11] corollaire 5.3, such π and π' have base changes Π and Π' to $GL_n(\mathbb{A}_L)$ such that for each place w of L above a place v of L^+, Π_w is the local base change of π_w. By strong multiplicity one for $GL_n(\mathbb{A}_L)$, $\Pi_w = \Pi'_w$ for each place w of L. Since each place of S is split in L and π and π' are assumed U_v-spherical at places $v \notin S$, we deduce that $\pi \cong \pi'$ as representations of $G(\mathbb{A}_{L^+})$. But by [Lab11] Théorème 5.4, π appears with multiplicity one in $\mathcal{A}$, so that $\pi = \pi'$. $\square$

In a similar vein, suppose that Π is a regular algebraic conjugate self-dual cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ which is unramified outside of places dividing S, and let U^S be as above. The following is also a consequence of [Lab11] Théorème 5.4, Corolllaire 5.3, and strong multiplicity one:

Lemma 5.5. *There is an automorphic representation π of $G(\mathbb{A}_{L^+})$ such that, at each finite place $v \notin S$, $\pi_v^{U_v} \neq 0$ and Π_v is the spherical base change of π_v (relative to our chosen hyperspecial maximal compact U_v, if v is inert).* $\square$

5.2. Galois representations.

5.2.1. Recall some notation from [BLGGT14] section 1.1. Let $\mathcal{G}_n$ be the algebraic group $(GL_n \times GL_1) \rtimes \{1, j\}$ where $j(g, a)j^{-1} = (a \, {}^t g^{-1}, a)$, $\mathcal{G}_n^0$ be the connected component $GL_n \times GL_1$ of $\mathcal{G}_n$, and $\nu : \mathcal{G}_n \to GL_1$ be defined by $\nu(g, a) = a$, $\nu(j) = -1$. If Γ is a group, Δ is an index 2 subgroup, and $\rho : \Gamma \to \mathcal{G}_n(A)$ is a representation (for some ring A) such that $\rho^{-1}(\mathcal{G}_n^0(A)) = \Delta$, then let $\breve{\rho}$ be the composition of $\rho|_\Delta$ with the projection $\mathcal{G}_n^0(A) \to GL_n(A)$.

5.2.2. *Ordinary deformations.* Suppose that $k/\mathbb{Q}_l$ is a finite extension and that E contains the images of all embeddings $k \hookrightarrow \overline{E}$. If $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(k,E)}$, and $\overline{r}_l : G_k \to GL_n(\mathbb{F})$ is a continuous representation, denote by $R_{\lambda,\mathrm{cr\text{-}ord}}^{\square}(\overline{r}_l)$ the ring called $R^{\triangle_\lambda,\mathrm{cr}}$ in [Ger10].

Proposition 5.6. *The scheme $\mathrm{Spec}\, R_{\lambda,\mathrm{cr\text{-}ord}}^{\square}(\overline{r}_l)$ is reduced, $\mathcal{O}$-flat and equidimensional of relative dimension $[k : \mathbb{Q}_l]\frac{n(n-1)}{2} + n^2$ over $\mathcal{O}$ (if it is non-zero). The $\overline{E}$-points of $\mathrm{Spec}\, R_{\lambda,\mathrm{cr\text{-}ord}}^{\square}(\overline{r}_l)[1/l]$ are those $\overline{E}$-points x of $\mathrm{Spec}\, R^{\square}(\overline{r}_l)[1/l]$ such that the associated Galois representation $r_{l,x}$ is ordinary of weight λ (in the sense of [Ger10] Definition 3.3.1) and crystalline.*

Proof. This can all be found in [Ger10] section 3.3. $\square$

Lemma 5.7. *If $k = \mathbb{Q}_l$, $\overline{r}_l$ is trivial and $\lambda = ((l-2)(n-1), (l-2)(n-2), \dots, (l-2), 0)$, then $R_{\lambda,\mathrm{cr\text{-}ord}}^{\square}(\overline{r}_l)$ is irreducible and non-zero.*

Proof. The representation $V = \bigoplus_{i=1}^n \mathcal{O}(-(i-1)(l-1))$ is a lift of $\overline{r}_l$ such that $V \otimes E$ is crystalline and ordinary of weight λ. The irreducibility is given by [Ger10] Lemma 3.4.3. $\square$

5.2.3. *Global deformations.* Suppose that l' is a prime, $L_v/\mathbb{Q}_{l'}$ is a finite extension, $\overline{r}_v : G_{L_v} \to GL_n(\mathbb{F})$ is a continuous representation and $\mathcal{C}_v$ is a finite set of irreducible components of $\mathrm{Spec}\, R^{\square}(\overline{r}_v)$ (if $l' \neq l$) or of $R_{\lambda,\mathrm{cr\text{-}ord}}^{\square}(\overline{r}_v)$ for some λ (if $l' = l$). Then by [BLGGT14] Lemma 1.2.2, $\mathcal{C}_v$ determines a local deformation problem for $\overline{r}_v$.

We recall some notation for global deformation problems from [CHT08], section 2.3. Suppose that:

- L is an imaginary CM field with maximal totally real subfield L^+;
- T is a finite set of places of L^+ which split in L containing all places above l;
- $\tilde{T}$ is a finite set of places of L consisting of exactly one place $\tilde{v}$ above each $v \in T$;
- $\overline{\rho} : G_{L^+} \to \mathcal{G}_n(\mathbb{F})$ is a continuous representation, unramified outside T, with $\overline{\rho}^{-1}(\mathcal{G}_n^0(\mathbb{F})) = G_L$;
- $\mu : G_{L^+} \to \mathcal{O}^\times$ is a continuous lift of $\nu \circ \overline{\rho}$;
- for each $v \in T$, $\mathcal{C}_v$ is a non-empty set of components of $R^{\square}(\breve{\overline{\rho}}|_{G_{L_{\tilde{v}}}})$ (if $v \nmid l$) or of some $R_{\lambda_v,\mathrm{cr\text{-}ord}}^{\square}(\breve{\overline{\rho}}|_{G_{L_{\tilde{v}}}})$ (if $v \mid l$).

Then the data

$$\mathcal{S} = (L/L^+, T, \tilde{T}, \mathcal{O}, \overline{\rho}, \mu, \{\mathcal{C}_v\}_{v \in T})$$

determines a deformation problem for $\overline{\rho}$; if $\breve{\overline{\rho}}$ is absolutely irreducible, then there is a universal deformation ring $R_{\mathcal{S}}^{\mathrm{univ}}$ and universal deformation

$$r_{\mathcal{S}}^{\mathrm{univ}} : G_{L^+} \to \mathcal{G}_n(R_{\mathcal{S}}^{\mathrm{univ}})$$

of type $\mathcal{S}$, defined in [CHT08] section 2.3.

Proposition 5.8. *If $\mu(c_v) = -1$ for all $v \mid \infty$ (where c_v is complex conjugation associated to v) then*

$$\dim R_{\mathcal{S}}^{\mathrm{univ}} \geq 1.$$

Proof. This follows from [CHT08] Corollary 2.3.5 and the dimension formulae for the $\mathcal{C}_v$; see [BLGGT14] Proposition 1.5.1. $\square$

Define also the T-framed deformation ring $R_{\mathcal{S}}^{\square_T}$ as in [CHT08] Proposition 2.2.9; it is an algebra over $\widehat{\bigotimes}_{v \in T} R_{\mathcal{C}_v}^{\square}$ where $R_{\mathcal{C}_v}^{\square}$ is the quotient of $R^{\square}(\breve{\overline{\rho}}|_{G_{L_{\tilde{v}}}})$ corresponding to $\mathcal{C}_v$.

5.2.4.   Now let L, λ, T, U and M be as in section 5.1.3, and suppose that M is finitely generated as an $\mathcal{O}$-module. Let $\mathfrak{m}$ be an ordinary maximal ideal of $\mathbb{T}_{\lambda}^T(U, M)$.

Proposition 5.9. *There is a unique continuous homomorphism*

$$r_{\mathfrak{m}} : G_{L^+, T} \to \mathcal{G}_n(\mathbb{T}_{\lambda}^T(U, M)_{\mathfrak{m}})$$

such that

(1) *$r_{\mathfrak{m}}^{-1}(\mathcal{G}_n^0(\mathbb{T}_{\lambda}^T(U, M)_{\mathfrak{m}})) = G_{L,T}$;*
(2) *$\nu \circ r_{\mathfrak{m}} = \epsilon^{1-n} \delta_{L/L^+}^n$;*
(3) *if $v \notin T$ splits as ww^c in L, then $r_{\mathfrak{m}}(\mathrm{Frob}_w)$ has characteristic polynomial*

$$\sum_{j=0}^n (-1)^j \mathrm{Nm}(w)^{j(j-1)/2} T_w^{(j)} X^{n-j};$$

(4) *for each $v \in S_l$, $r_m|_{G_{L_v}}$ factors through $R_{\lambda, \mathrm{cr\text{-}ord}}^{\square}(\overline{r}_{\mathfrak{m}}|_{G_{L_{\tilde{v}}}})$.*

Proof. Suppose first that M is finite free as an $\mathcal{O}$-module. Then the construction of $r_{\mathfrak{m}}$ is standard (see [CHT08] Proposition 3.4.4.). The first three properties are deduced as in that reference, and the final property is proved as in [Ger10] Lemma 3.3.4.

For general M, if we can show that M admits a surjection from an $\mathcal{O}[U]$-module P that is finite free as an $\mathcal{O}$-module, then we are done by the previous case. But the action of U on M factors through a finite quotient $\overline{U}$ of U, and we may take P to be the projective envelope of M as an $\mathcal{O}[\overline{U}]$-representation. $\square$

5.3. **Realising local representations globally.** Recall that we have a representation $\overline{\rho} : G_F \to GL_n(\mathbb{F})$. The aim of this section is to globalise $\overline{\rho}$, as in Proposition 5.12 below. We follow [EG14] Appendix A closely, and the reader wishing to follow the arguments will need to have that paper to hand. Note that in [EG14] the residue characteristic of the coefficient field is called p, whereas here it is called l.

5.3.1. *Adequacy.* Thorne, in [Tho15] Definition 2.20) has modified the definition of adequacy from that in [Tho12] to allow some cases where $l \mid n$ — the definitions coincide if $l \nmid n$. Let us repeat the new definition here:

Definition 5.10. Let V be a finite-dimensional vector space over $\overline{\mathbb{F}}$. A subgroup $H \subset GL(V)$ is **adequate** if it acts irreducibly on V and if:

(1) $H^1(H, \overline{\mathbb{F}}) = 0$;
(2) $H^1(H, \mathrm{End}(V)/\overline{\mathbb{F}}) = 0$ where H acts on $\mathrm{End}(V)$ by conjugation and $\overline{\mathbb{F}}$ is the subspace of scalar endomorphisms;
(3) For each simple $\overline{\mathbb{F}}[H]$-submodule $W \subset \mathrm{End}(V)$, there is a semisimple element $\sigma \in H$ with an eigenvalue $\alpha \in \mathbb{F}$ such that $\mathrm{tr}\, e_{\sigma,\alpha} W \neq 0$, where $e_{\sigma,\alpha}$ is the projection onto the α-eigenspace of σ.

With this definition, the main theorems of [Tho12] (Theorems 7.1, 9.1, 10.1 and 10.2) continue to hold, by [Tho15] Corollary 7.3.

Lemma 5.11. *Let $GL_n.2$ be the smallest algebraic subgroup of GL_{2n} containing the block diagonal matrices of the form $(g, {}^tg^{-1})$ and a matrix J such that $J(g, {}^tg^{-1})J^{-1} = ({}^tg^{-1}, g)$. Then for m sufficiently large, both*

$$(GL_n.2)(\mathbb{F}_{l^m}) \subset GL_{2n}(\overline{\mathbb{F}}_l)$$

and

$$GL_n(\mathbb{F}_{l^m}) \subset GL_n(\overline{\mathbb{F}}_l)$$

are adequate. In other words, Lemma A.1 of [EG14] continues to hold with the revised definition of adequate.

Proof. This is a consequence of [GHT14] Theorem 11.5, remembering our running assumption that $l > 2$. $\qquad\square$

5.3.2. The main result of this section is:

Proposition 5.12. *There is an imaginary CM field L with maximal totally real subfield L^+, and there are continuous representations*

$$\overline{r} : G_{L^+} \to \mathcal{G}_n(\overline{\mathbb{F}})$$

and

$$r : G_{L^+} \to \mathcal{G}_n(\overline{E})$$

satisfying the following hypotheses:

(1) *r is a lift of $\overline{r}$;*
(2) *$\overline{r}^{-1}(\mathcal{G}_n^0(\overline{\mathbb{F}})) = G_L$;*
(3) *$\breve{r}$ is of the form $r_{l,\iota}(\pi, \chi)$ for a regular algebraic, cuspidal, polarized automorphic representation (π, χ) (see [BLGGT14], Theorem 2.1.1 for the notation $r_{l,\iota}$);*
(4) *$\breve{r}(G_{L(\zeta_l)}) = GL_n(\mathbb{F}_{l^m})$ for m large enough that the conclusion of Lemma 5.11 holds (in particular, $\breve{r}(G_{L^+(\zeta_l)})$ is adequate);*
(5) *$\nu \circ r = \epsilon^{1-n}\delta_{L/L^+}^n$ and similarly for $\overline{r}$ (note that this determines χ);*
(6) *Every place v of L^+ dividing lp splits completely in L;*
(7) *For each place v of L^+ dividing p, there is an isomorphism $L_v^+ \cong F$ and a place $\tilde{v}$ of L dividing v such that $\breve{r}|_{G_{L_{\tilde{v}}}} \cong \overline{\rho}$;*

(8) *For each place v of L^+ dividing l, we have that $L_v^+ = \mathbb{Q}_l$ and there is a place $\tilde{v}$ of L dividing v such that $\breve{\bar{r}}|_{G_{L_{\tilde{v}}}}$ is trivial and $\breve{r}|_{G_{L_{\tilde{v}}}}$ is ordinary of weight λ for λ as in Lemma 5.7;*

(9) $\overline{L}^{\ker \bar{r}}$ *does not contain $L(\zeta_l)$;*

(10) *if v is a place of L^+ not dividing lp, then $\bar{r}$ and r are unramified at v;*

(11) *$[L^+ : \mathbb{Q}]$ is divisible by 4, and L/L^+ is unramified at all finite places.*

The first step is to realise $\bar{r}$ as the local component of some (not yet automorphic) representation $\bar{r}$, using [Cal12] Proposition 3.2.

Lemma 5.13. *There exist a CM field L_1 with maximal totally real subfield L_1^+ and a continuous representation $\bar{r} : G_{L_1^+} \to \mathcal{G}(\overline{\mathbb{F}})$ satisfying properties 2 and 4–11 of Proposition 5.12 (at least as they pertain to $\bar{r}$).*

Proof. This is a straightforward modification of the proof of [EG14] Proposition A.2 to include conditions on L_1 and $\bar{r}$ at places dividing p. $\qquad\square$

5.3.3.   Now we show that $\bar{r}$ is potentially automorphic over some CM extension L/L_1. This basically follows the proof of Proposition 3.3.1 of [BLGGT14], making modifications to control the splitting in L of places of L_1 above l and p (as in [EG14]). The first step is to show that this $\bar{r}$ lifts to a characteristic zero representation with good properties.

Lemma 5.14. *Let $\bar{r}$ be as in Lemma 5.13. Then there is a continuous representation $r : G_{L_1^+} \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$ lifting $\bar{r}$ satisfying all of the properties of Proposition 5.12 except possibly automorphy (property 3).*

Proof. This is proved in [BLGGT14], Proposition 3.2.1, under the hypothesis that $l \geq 2n + 1$. We examine the proof of that proposition and show that in our case we may remove the hypothesis on l. The only way in which this hypothesis is used is to verify, using Proposition 2.1.2 of that paper, the adequacy of the image of the induction of $\breve{\bar{r}}$ from $G_{L_1(\zeta_l)}$ to $G_{L_1^+(\zeta_l)}$. However, by property 4 of Proposition 5.12 we can use Lemma 5.11 instead of [BLGGT14] Proposition 2.1.2. (Note that Theorems 9.1 and 10.2 of [Tho12] remain true with this definition, and so in [BLGGT14], Theorems 2.4.1 and 2.4.2, and hence also Proposition 3.2.1 remain true.) $\qquad\square$

Lemma 5.15. *There is a CM extension L/L_1, linearly disjoint from $\overline{L_1}^{\ker \bar{r}}(\zeta_l)$ over L_1, such that every place of L_1 dividing lp splits completely in L and such that L and $\breve{r}|_{G_L}$ satisfy all the properties required in Proposition 5.12. In particular, Proposition 5.12 is true.*

Proof. The proof of [EG14] Proposition A.6 goes through with the following modifications – we temporarily adopt the notation of their proof to indicate what must be changed. The field $(L')^+$ must be chosen so that, for each place $v \mid p$ of $(L')^+(\zeta_N)^+$, there is a point $P_v \in \tilde{T}((L')^+(\zeta_N)_v^+)$. The field extension F^+/L^+ can then be chosen so that all the places of L^+ above p split completely (as well as all those above l). Finally, instead of using Theorem 4.2.1 of [BLGGT14] we use Theorem 2.4.1 of that paper, which applies by our assumption that r is ordinary. $\qquad\square$

5.4. **Patching.**

5.4.1. Let $\bar{r} : G_{L^+} \to \mathcal{G}_n(\overline{\mathbb{F}})$ be the representation provided by Proposition 5.12, and (enlarging E if necessary) assume that $\bar{r}$ is valued in $\mathcal{G}_n(\mathbb{F})$. Thus $\bar{r}$ is the reduction modulo λ of the Galois representation $r_{l,\iota}(\pi, \chi)$ associated to some regular algebraic polarized cuspidal automorphic representation (π, χ) of $GL_n(\mathbb{A}_L)$. Choose a place v_1 of L^+ that splits as $\tilde{v}_1 \tilde{v}_1^c$ in L so that every lift of $\bar{r}|_{G_{L_{\tilde{v}_1}}}$ is unramified (and so $R^\square(\breve{\bar{r}}|_{G_{L_{\tilde{v}_1}}})$ is equal to the unramified deformation ring, and is in particular formally smooth). Take S to be the set of places of L^+ dividing p together with the place v_1, and recall that $T = S \cup S_l$ and $\tilde{T}$ is a choice of a place $\tilde{v}$ of L above each $v \in T$. Let $\lambda \in (\mathbb{Z}_n^+)^{\tilde{I}_l}$ have all components equal to the weight in Lemma 5.7. Let $U = \prod_v U_v$ where:

- for v a place of L^+ split in L, $U_v = G(\mathcal{O}_{L_v^+})$;
- for v a place of L^+ inert in L, $U_v \subset GL_n(L_v^+)$ is a hyperspecial maximal compact subgroup;
- for $v = v_1$, U_{v_1} is the preimage under $\iota_{\tilde{v}_1}$ of the Iwahori subgroup of $GL_n(\mathcal{O}_{L_v^+})$.

For $v \in T$ a place of L^+ dividing $\tilde{v} \in \tilde{T}$, let $R_{\tilde{v}}$ be:

- $R_{\tilde{v}} = R^\square(\breve{\bar{r}}|_{G_{L_{\tilde{v}}}})$ if $v \in S$;
- $R_{\tilde{v}} = R^\square_{\lambda,\text{cr-ord}}(\breve{\bar{r}}|_{G_{L_{\tilde{v}}}})$ if $v \in S_l$.

Then for $v \in S_l$ or $v = v_1$, $R_{\tilde{v}}$ is a domain (by Lemma 5.7 in the former case). Let $R^{\text{loc}} = \widehat{\bigotimes}_{v \in T} R_{\tilde{v}}$.

There is a global deformation problem

$$\mathcal{S} = (L/L^+, T, \tilde{T}, \mathcal{O}, \bar{r}, \epsilon^{1-n} \delta^n_{L/L^+}, \{R_{\tilde{v}}\}_{v \in T})$$

with universal deformation $r^{\text{univ}}_{\mathcal{S}} : G_{L^+,T} \to \mathcal{G}_n(R^{\text{univ}}_{\mathcal{S}})$. Let $f_\pi : \mathbb{T}^T \to \mathcal{O}$ be the homomorphism such that $\iota \circ f_\pi(T_w^{(j)})$ gives the eigenvalue of $T_w^{(j)}$ acting on π_w via ι_w, and let $\mathfrak{m}$ be the kernel of the composite $(\mathcal{O} \twoheadrightarrow \mathbb{F}) \circ f_\pi$.

5.4.2. Let $U_p = \prod_{v|p} G(\mathcal{O}_{L_v^+})$ and $U_S = \prod_{v \in S} U_v = U_p U_{v_1}$. Let $\mathcal{R}$ be the category of smooth representations of U_p on finitely generated $\mathcal{O}$-modules and let $\mathcal{R}^f$ be the category of smooth representations of U_p on finite-length $\mathcal{O}$-modules. If $\sigma \in \mathcal{R}$ then let M_σ be the underlying module of σ regarded as a representation of U_S by letting U_p act through σ and U_{v_1} act trivially. We define an $R^{\text{univ}}_{\mathcal{S}}$-algebra $\mathbb{T}(\sigma)$ and a $\mathbb{T}(\sigma)$-module $H(\sigma)$ by:

- $\mathbb{T}(\sigma) = \mathbb{T}_\lambda(U, M_\sigma)_{\mathfrak{m}}$ with the $R^{\text{univ}}_{\mathcal{S}}$-algebra structure provided by Proposition 5.9;
- $H(\sigma) = S_\lambda(U, M_\sigma)_{\mathfrak{m}}$.

5.4.3. By Lemma 5.11 and property (4) of Proposition 5.12, $\breve{\bar{r}}|_{G_{L(\varsigma_l)}}$ is adequate. Using Proposition 4.4 of [Tho12] and following the proof of his Theorem 6.8, we obtain an integer $r \geq [L^+ : \mathbb{Q}]\frac{n(n-1)}{2}$ and, for each $N \geq 1$, a set Q_N, disjoint from T, of r finite places of L^+ split in L and a set $\tilde{Q}_N$ of choices of places of L above those of Q_N. As in [Tho12], for each N and each σ we can find rings R^{univ}_N and $R^{\square_T}_N$, an R^{univ}_N-algebra $\mathbb{T}_N(\sigma)$ and a finitely generated $\mathbb{T}_N(\sigma)$-module $H_N(\sigma)$ enjoying the following properties:

- There is an isomorphism $R^{\text{univ}}_N \hat{\otimes} \mathcal{O}[[y_1, \ldots, y_{n^2 \# T}]] \cong R^{\square_T}_N$.

- For each $v \in Q_N$, $\mathrm{Nm}\, v \equiv 1 \mod l^N$. Let Δ_N be the maximal l-power-order quotient of $\kappa(\tilde{v})^\times$, and let $\mathfrak{a}_N$ be the augmentation ideal in the group ring $\mathcal{O}[\Delta_N]$.
- There are natural homomorphisms $\mathcal{O}[\Delta_N] \to R_N^{\mathrm{univ}}$ and $\mathcal{O}[\Delta_N] \to \mathrm{End}(H_N(\sigma))$ such that the composite $R_N^{\mathrm{univ}} \to \mathbb{T}_N(\sigma) \to \mathrm{End}(H_N(\sigma))$ is an $\mathcal{O}[\Delta_N]$-algebra homomorphism.
- With the above $\mathcal{O}[\Delta_N]$-algebra structures, there are natural isomorphisms $R_N^{\mathrm{univ}}/\mathfrak{a}_N \xrightarrow{\sim} R_S^{\mathrm{univ}}$, $\mathbb{T}_N(\sigma)/\mathfrak{a}_N \xrightarrow{\sim} \mathbb{T}(\sigma)$, and $H_N(\sigma)/\mathfrak{a}_N \xrightarrow{\sim} H(\sigma)$ (this relies on Lemma 5.2).
- The map $\mathcal{O}[\Delta_N] \to R_N^{\mathrm{univ}} \to \mathbb{T}_N(\sigma)$ makes $H_N(\sigma)$ into a finite free $\mathcal{O}[\Delta_N]$-module.
- We may and do choose a surjective $\mathcal{O}$-algebra homomorphism
$$R^{\mathrm{loc}}[[z_1, \ldots, z_g]] \twoheadrightarrow R_N^{\square_T}$$
  where $g = r - [L^+ : \mathbb{Q}]\frac{n(n-1)}{2}$.
- The functor $\sigma \mapsto H_N(\sigma)$ is a covariant exact functor from $\mathcal{R}$ to the category of finitely generated R_N^{univ}-modules.

Remark 5.16. Strictly speaking, the proof in [Tho12] that
$$R_N^{\mathrm{univ}} \to \mathrm{End}(H_N(\sigma))$$
is an $\mathcal{O}[\Delta]$-algebra homomorphism, and the construction of the isomorphism
$$H_N(\sigma)/\mathfrak{a}_N \to H(\sigma),$$
require that σ be finite free as an $\mathcal{O}$-module (to apply Propositions 5.9 and 5.12 in that paper). However, we can remove this constraint by writing σ as a quotient of a U_p-representation that *is* finite free as an $\mathcal{O}$-module, as in the proof of Proposition 5.9.

Write $H_N^{\square_T}(\sigma) = H_N(\sigma) \otimes_{R_N^{\mathrm{univ}}} R_N^{\square_T}$. We pick isomorphisms
$$R_N^{\square_T} \xrightarrow{\sim} R_N^{\mathrm{univ}} \hat{\otimes} \mathcal{O}[[y_1, \ldots, y_{n^2 \# T}]]$$
and
$$R_S^{\square_T} \xrightarrow{\sim} R_S^{\mathrm{univ}} \hat{\otimes} \mathcal{O}[[y_1, \ldots, y_{n^2 \# T}]]$$
compatible with reduction modulo $\mathfrak{a}_N$. Let
$$R_\infty = R^{\mathrm{loc}}[[z_1, \ldots, z_g]]$$
and
$$S_\infty = (\varprojlim \mathcal{O}[\Delta_N]) \hat{\otimes} \mathcal{O}[[y_1, \ldots, y_{n^2 \# T}]]$$
$$\cong \mathcal{O}[[x_1, \ldots, x_r, y_1, \ldots, y_{n^2 \# T}]]$$
and note that (by Theorem 2.5 and Proposition 5.6) we have
$$\dim R_\infty = 1 + n^2 \# T + [L^+ : \mathbb{Q}]\frac{n(n-1)}{2} + r - [L^+ : \mathbb{Q}]\frac{n(n-1)}{2}$$
$$= \dim S_\infty.$$

Write $\mathfrak{a}$ for the kernel of the map $S_\infty \to \mathcal{O}$ taking x_i and y_i to zero. Thus $R_N^{\square_T}/\mathfrak{a} \xrightarrow{\sim} R_S^{\mathrm{univ}}$ and $H_N^{\square_T}(\sigma)/\mathfrak{a} \xrightarrow{\sim} H(\sigma)$.

5.4.4. We patch the modules $H_N^{\square_T}(\sigma)$ following the proof of the sublemma in [BLGG11], Theorem 3.6.1. Pick representations $\sigma_1, \sigma_2, \ldots$ such that each of the countably many isomorphism classes in $\mathcal{R}^f$ is represented by exactly one σ_i. For $h \in \mathbb{N}$, let $\mathcal{R}^f_{\leq h}$ be the full subcategory of $\mathcal{R}^f$ whose objects are $\sigma_1, \ldots, \sigma_h$.

Choose a strictly increasing sequence $(h(N))_N$ of positive integers. Let $\mathfrak{c}_N = \ker(S_\infty \to \mathcal{O}[\Delta_N] \hat{\otimes} \mathcal{O}[[y_1, \ldots, y_{n^2 \# T}]])$ and choose a sequence $\mathfrak{b}_1 \supset \mathfrak{b}_2 \supset \ldots$ of open ideals of S_∞ such that $\mathfrak{b}_N \supset \mathfrak{c}_N$ for all N and $\bigcap_N \mathfrak{b}_N = (0)$. Choose also open ideals $\mathfrak{d}_1 \supset \mathfrak{d}_2 \supset \ldots$ of $R_{\mathcal{S}}^{\mathrm{univ}}$ with $\mathfrak{b}_N R_{\mathcal{S}}^{\mathrm{univ}} + \ker(R_{\mathcal{S}}^{\mathrm{univ}} \to \mathbb{T}(\sigma)) \supset \mathfrak{d}_N \supset \mathfrak{b}_N R_{\mathcal{S}}^{\mathrm{univ}}$ for all $\sigma \in \mathcal{R}^f_{\leq h(N)}$ and $\bigcap_N \mathfrak{d}_N = (0)$.

Define a **patching datum of level** N to be:

- a surjective $\mathcal{O}$-algebra homomorphism

$$\phi : R_\infty \twoheadrightarrow R_{\mathcal{S}}^{\mathrm{univ}}/\mathfrak{d}_N;$$

- a covariant, exact functor $\mathcal{M}_N$ from $\mathcal{R}^f_{\leq h(N)}$ to the category of $R_\infty \hat{\otimes} S_\infty$-modules that are finite free over $S_\infty/\mathfrak{b}_N$;
- for $\sigma \in \mathcal{R}^f_{\leq h(N)}$, functorial isomorphisms of R_∞-modules

$$\mathcal{M}_N(\sigma)/\mathfrak{a} \xrightarrow{\sim} H(\sigma)/\mathfrak{b}_N$$

(the right hand side being an R_∞-module via ϕ).

Since $S_\infty/\mathfrak{b}_N$, $R_{\mathcal{S}}^{\mathrm{univ}}/\mathfrak{d}_N$, $H(\sigma)/\mathfrak{d}_N$ are finite sets and the sets of objects and morphisms in $\mathcal{R}^f_{\leq h(N)}$ are finite, there are only finitely many patching data of level N. Note that if $N' \geq N$ then from any patching datum of level N' we can get one of level N by reducing modulo $\mathfrak{b}_N$ and $\mathfrak{d}_N$ and restricting $\mathcal{M}_{N'}$ to $\mathcal{R}^f_{h(N)}$.

For each pair of integers $M \geq N \geq 1$ define a patching datum $D(M, N)$ of level N by taking:

- $\phi : R_\infty \twoheadrightarrow R_N^{\square_T} \twoheadrightarrow R/\mathfrak{d}_N$ where the first map is our chosen presentation of $R_N^{\square_T}$ over R^{loc} and the second is induced by $R_N^{\square_T}/\mathfrak{a} \xrightarrow{\sim} R_{\mathcal{S}}^{\mathrm{univ}}$;
- $\mathcal{M}_N(\sigma) = H_M^{\square_T}(\sigma)/\mathfrak{b}_N$, which is finite free over $S_\infty/\mathfrak{b}_N$ and is an R_∞-module via $R_\infty \twoheadrightarrow R_M^{\square_T} \twoheadrightarrow \mathbb{T}_M^{\square_T}$ (clearly $\mathcal{M}_N$ is a functor);
- the isomorphism $\psi : \mathcal{M}_N/\mathfrak{a} \xrightarrow{\sim} H(\sigma)/\mathfrak{b}_N$ coming from the natural isomorphism $H_M^{\square_T}(\sigma)/\mathfrak{a} \xrightarrow{\sim} H(\sigma)/\mathfrak{b}_N$.

Since there are only finitely many isomorphism classes of patching datum of each level N, we may choose an infinite sequence of pairs $(M_j, N_j)_{j \geq 1}$ with $M_j \geq N_j$, $M_{j+1} > M_j$ and $N_{j+1} > N_j$ such that $D(M_{j+1}, N_{j+1})$ reduces to $D(M_j, N_j)$ for each j. We may therefore define a functor H_∞ from $\mathcal{R}^f$ to the category of $R_\infty \hat{\otimes} S_\infty$-modules by the formula:

$$H_\infty(\sigma) = \varprojlim_j H_{M_j}^{\square_T}(\sigma)/\mathfrak{b}_{N_j}$$

(and extending to the whole of $\mathcal{R}^f$ by picking an isomorphism from each object to one of the σ_i). Note that the terms in the limit are defined for j sufficiently large. Extend H_∞ to $\mathcal{R}$ by setting $H_\infty(\varprojlim \sigma_i) = \varprojlim H_\infty(\sigma_i)$.

5.4.5. We need to verify that H_∞ has the properties needed for the proof of Theorem 4.6. The functor H_∞ is exact and covariant, and for all σ we have

$$H_\infty(\sigma \otimes_{\mathcal{O}} \mathbb{F}) = H_\infty(\sigma) \otimes \mathbb{F}$$

(these statements all follow from the corresponding statements at finite level).

Lemma 5.17. *For each σ, the support $\operatorname{supp}_{R_\infty}(H_\infty(\sigma))$ is a union of irreducible components of $\operatorname{Spec} R_\infty$.*

Proof. We may factor the map $S_\infty \to \operatorname{End}_{R_\infty}(H_\infty(\sigma))$ through a map $S_\infty \to R_\infty$ (since we may do this at finite level by definition of the action of S_∞). So we have a map $S_\infty \to R_\infty$ and a finitely generated R_∞-module $H_\infty(\sigma)$ that is finite free over the regular local ring S_∞. Thus we have:

$$\begin{aligned}
\operatorname{depth}_{R_\infty}(H_\infty(\sigma)) &\geq \operatorname{depth}_{S_\infty}(H_\infty(\sigma)) \\
&= \dim S_\infty \\
&= \dim R_\infty \\
&\geq \operatorname{depth}_{R_\infty}(H_\infty(\sigma)).
\end{aligned}$$

Therefore by [Tay08], Lemma 2.3, $\operatorname{supp}_{R_\infty}(H_\infty(\sigma))$ is a union of irreducible components of $\operatorname{Spec} R_\infty$. $\qquad\square$

The argument of the next lemma goes back to [Dia97]:

Lemma 5.18. *Let $\mathfrak{q}$ be a prime ideal of R_∞ such that $(R_\infty)_{\mathfrak{q}}$ is regular. Then $H_\infty(\sigma)_{\mathfrak{q}}$ is finite free over $(R_\infty)_{\mathfrak{q}}$.*

Proof. We may suppose that $\mathfrak{q} \in \operatorname{supp}_{R_\infty} H_\infty$. Since $(R_\infty)_{\mathfrak{q}}$ is regular, it is a domain. By the previous lemma, $(R_\infty)_{\mathfrak{q}}$ acts faithfully on $(H_\infty(\sigma))_{\mathfrak{q}}$. Thus $(R_\infty)_{\mathfrak{q}}$ is finite over $(S_\infty)_{S_\infty \cap \mathfrak{q}}$. The argument of the previous lemma now shows that

$$\operatorname{depth}_{(R_\infty)_{\mathfrak{q}}}(H_\infty(\sigma)_{\mathfrak{q}}) = \operatorname{depth}(R_\infty)_{\mathfrak{q}}.$$

The module $H_\infty(\sigma)_{\mathfrak{q}}$ has finite projective dimension over $(R_\infty)_{\mathfrak{q}}$ and the Auslander–Buchsbaum formula holds:

$$\operatorname{depth}_{(R_\infty)_{\mathfrak{q}}}(H_\infty(\sigma)_{\mathfrak{q}}) + \operatorname{pd}_{(R_\infty)_{\mathfrak{q}}}(H_\infty(\sigma)_{\mathfrak{q}}) = \operatorname{depth}(R_\infty)_{\mathfrak{q}}.$$

Therefore $H_\infty(\sigma)_{\mathfrak{q}}$ is a finitely generated projective $(R_\infty)_{\mathfrak{q}}$-module as required. $\square$

5.4.6.   Note that R_∞ is a completed tensor product of the ring

$$\widehat{\bigotimes_{v|p}} R^{\square}(\breve{\overline{r}}|_{G_L})$$

with a complete local noetherian domain, and therefore giving a minimal prime $\mathfrak{p}$ of R_∞ is the same as giving a minimal prime $\mathfrak{p}_v$ of each

$$R^{\square}(\breve{\overline{r}}|_{G_{L_v}}) \cong R^{\square}(\overline{\rho}).$$

Proposition 5.19. *Let $\sigma \in \mathcal{R}$ be finite free as an $\mathcal{O}$-module and of the form $\otimes_{v|p} \sigma_v$ for representations σ_v of $U_v \cong GL_n(\mathcal{O}_F)$. For each place v of L_v^+ above p let τ_v be an inertial type and pick a minimal prime $\mathfrak{p}_v$ of $R^{\square}(\overline{\rho}, \tau_v)$. Let $\mathfrak{p}$ be the minimal prime of R_∞ determined by the $\mathfrak{p}_v$. Then $H_\infty(\tau)/\mathfrak{p}$ is generically free of rank*

$$n! \prod_{v|p} m((\sigma_v \otimes \overline{E})^{\vee}, \tau_v)$$

over $R_\infty/\mathfrak{p}$.

Proof. Let $\mathcal{S}'$ be the deformation problem

$$(L/L^+, T, \tilde{T}, \mathcal{O}, \overline{r}, \epsilon^{1-n}\delta^n_{L/L^+}, \{R'_{\tilde{v}}\}_{v \in T})$$

where $R'_{\tilde{v}} = R_{\tilde{v}}$ unless $v|p$, in which case $R'_{\tilde{v}} = R_{\tilde{v}}/\mathfrak{p}_v$. Then $R^{\mathrm{univ}}_{\mathcal{S}'}$ is a quotient of $R^{\mathrm{univ}}_{\mathcal{S}}$.

By Proposition 5.8,

$$\dim R^{\mathrm{univ}}_{\mathcal{S}'} \geq 1.$$

By [Tho12], Theorem 10.2, $R^{\mathrm{univ}}_{\mathcal{S}'}$ is a finite $\mathcal{O}$-module; it therefore admits an $\mathcal{O}$-algebra homomorphism

$$x : R^{\mathrm{univ}}_{\mathcal{S}'} \to \mathcal{O}'$$

for a finite extension $\mathcal{O}'/\mathcal{O}$; enlarging E, we may assume that $\mathcal{O}' = \mathcal{O}$. There is a corresponding representation $r' : G_{L^+} \to \mathcal{G}_n(\mathcal{O})$. By [Tho12], Theorem 9.1, r' is the representation attached to some regular algebraic polarized cuspidal automorphic representation $(\pi', \delta^n_{L/L^+})$ of $GL_n(\mathbb{A}_L)$ with $\pi = \otimes_v \pi_v$. By Lemmas 5.4 and 5.5 we see that the fibre of $H(\sigma)$ at x (for any $\sigma = \bigotimes_{v|p} \sigma_v$) has dimension:

$$\dim \mathrm{Hom}_{U_S}\left((\sigma \otimes \overline{E})^\vee, \bigotimes_{v \in S} \pi_v \otimes_{\mathbb{C}, \iota^{-1}} \overline{E}\right)$$

$$= \dim \pi^{U_v}_{\tilde{v}_1} \prod_{v|p} \dim \mathrm{Hom}_{U_v}\left((\sigma_v \otimes \overline{E})^\vee, \pi_{\tilde{v}} \otimes_{\mathbb{C}, \iota^{-1}} \overline{E}\right)$$

$$= n! \prod_{v|p} m((\sigma_v \otimes \overline{E})^\vee, \tau_v).$$

The last equality results from the fact that, by local–global compatibility, for $v \mid p$ each $\pi_{\tilde{v}} \otimes_{\mathbb{C}, \iota^{-1}} \overline{E}$ is a tempered (and hence generic) representation of type τ_v (see [BLGGT14] Theorem 2.1.1, for example). The factor of $n!$ is the contribution from the Iwahori invariants in the unramified principal series representation $\pi_{\tilde{v}_1}$.

Now choose an $\mathcal{O}$-point $\tilde{x}$ of $\mathrm{Spec}\, R^\infty$ above x. As $\tilde{x}$ is (in the terminology of Proposition 3.6) a non-degenerate point of each factor $R^\square_v$ of R_∞, we see that $\mathrm{Spec}\, R_\infty$ is formally smooth at $\tilde{x}$. By Lemma 5.18, we see that $H_\infty(\sigma)_{\tilde{x}}$ is free over $(R_\infty)_{\tilde{x}}$. To determine the rank, note that $H_\infty(\sigma)_{\tilde{x}}/\mathfrak{a} = H(\sigma)_x$, and applying the above calculation we get the proposition. $\qquad\square$

We have therefore shown that H_∞ has all the properties needed for the proof of Theorem 4.6. To be specific, in the notation of that proof we take d equal to the number of places v of L^+ dividing p, $c = n!$, and $A = \widehat{\bigotimes}_{v \in T, v \nmid p} R_{\tilde{v}}[[z_1, \ldots, z_g]]$.

6. K-TYPES.

We give a modification of the construction in [SZ99] of the representations $\sigma(\tau)$ that allows us to analyze their reduction modulo l, and prove the formula for $m(\sigma(\tau), \tau')$ alluded to in Proposition 4.3.

6.1. **Symmetric groups.** If $P \in \mathrm{Part}$ with $\deg P = n$, for each $i \in \mathbb{N}$ let $T_{P,i}$ be the subset

$$\left\{1 + \sum_{j=1}^{i-1} P(j), \ldots, \sum_{j=1}^{i} P(j)\right\}$$

of $\{1, \ldots, n\}$. Let S_P be the subgroup of S_n stabilising each $T_{P,i}$, so that $S_P = \prod_i S_{P(i)}$. Let

$$\pi_P^\circ = \mathrm{Ind}_{S_P}^{S_n}(\mathrm{sgn})$$

where sgn is the sign representation.

Definition 6.1. Let σ_P° be the unique irreducible representation of S_n that appears in π_P° and that appears in no $\pi_{P'}^\circ$ for $P' \succ P$.

Every irreducible representation is of the form σ_P° for a unique P. Note that this is not the standard association of representations of S_n to partitions, but rather its twist by the sign representation.

Definition 6.2. The **Kostka number** $m(P, P')$ is the multiplicity with which σ_P° appears in $\pi_{P'}^\circ$.

We adopt the conventions that if $\deg P \neq \deg P'$ then $m(P, P') = 0$, while if $\deg P = \deg P' = 0$ then $m(P, P') = 1$. Thus $m(P, P') > 0$ if and only if $P \succeq P'$, and if $P = P'$ then $m(P, P') = 1$. This *does* coincide with the standard definition of Kostka numbers.

6.2. **PSH-algebras.** For our calculations of multiplicities we will require the notion of a PSH-algebra, due to Zelevinsky [Zel81]; see also chapter 3 of [GR14] and the proof of the Proposition in [SZ99] section 4.

Definition 6.3. A **positive self-adjoint Hopf** (or **PSH-**) **algebra** is a graded connected Hopf algebra

$$R = \bigoplus_{n \geq 0} R_n$$

over $\mathbb{Z}$, with multiplication $m : R \otimes R \to R$ and comultiplication $\mu : R \to R \otimes R$, together with a $\mathbb{Z}$-basis Σ of homogeneous elements with the following property: let $\langle \cdot, \cdot \rangle$ be the $\mathbb{Z}$-bilinear form on R making Σ an orthonormal basis. Then for all $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ we have

$$\langle m(\sigma_1 \otimes \sigma_2), \sigma_3 \rangle = \langle \sigma_1 \otimes \sigma_2, \mu(\sigma_3) \rangle > 0.$$

Suppose that R is a PSH-algebra, with notation as in the definition. An element $\sigma \in R$ is **primitive** if $\mu(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$. Say that R is **indecomposable** if there is a unique primitive element in Σ. The basic structure theorem is then:

Theorem 6.4. *(Zelevinsky) Let R be a PSH-algebra and Σ its distinguished basis. For each primitive $\sigma \in \Sigma$ there is an indecomposable sub–PSH-algebra $R(\sigma)$ of R such that*

$$\bigotimes_\sigma R(\sigma) \xrightarrow{\sim} R$$

is an isomorphism of PSH-algebras.[5]
If R and R' are indecomposable PSH-algebras then, after rescaling the gradings so that each has a primitive element of degree one, there are precisely two isomorphisms of PSH-algebras between R and R'.

---

[5]If there are infinitely many primitive elements of Σ, this should be interpreted as the direct limit of the tensor products over finite subsets of the primitive elements in Σ.

We can obtain an indecomposable PSH-algebra R^S from the representation theory of the symmetric group as follows: let R_n^S be the Grothendieck group of representations of S_n, and take Σ to be the subset of isomorphism classes of irreducible representations. The multiplication is given by induction: if σ_1 and σ_2 are irreducible representations of degrees S_{n_1} and S_{n_2} then

$$m(\sigma_1 \otimes \sigma_2) = \mathrm{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}}(\sigma_1 \otimes \sigma_2)$$

regarded as an element of the Grothendieck group. Similarly the comultiplication is given by restriction: if σ is a representation of S_n then

$$\mu(\sigma) = \sum_{a+b=n} \mathrm{Res}_{S_a \times S_b}^{S_n} \sigma$$

where we have identified the Grothendieck group of representations of $S_a \times S_b$ with the tensor product of those of S_a and S_b. That this (with the obvious unit and counit) is a Hopf algebra is an exercise using Mackey's theorem (see [GR14] Corollary 4.26), and the self-adjointness property is a consequence of Frobenius reciprocity. The unique primitive element is the trivial representation of the trivial group. The non-identity isomorphism $R^S \to R^S$ takes the trivial representation of any S_n to the sign representation.

6.3. **Finite general linear groups.** Let k be a finite field, $n \geq 1$ be an integer, and $\overline{G} = GL_n(k)$. For all unsupported assertions in this subsection see [SZ99] §4.

Definition 6.5. Let $\overline{\mathcal{I}}_0$ be the union over all d of the set of isomorphism classes of cuspidal representations of $GL_d(k)$. Let $\overline{\mathcal{I}}$ be the set of functions $\overline{\mathcal{P}} : \overline{\mathcal{I}}_0 \to \mathrm{Part}$ with finite support.

The degree $\deg \overline{\mathcal{P}}$ of an element of $\overline{\mathcal{I}}$ is defined to be the sum

$$\sum_{\sigma \in \overline{\mathcal{I}}_0} \deg(\overline{\mathcal{P}}(\sigma)) \dim \sigma.$$

Every irreducible representation of $\overline{G}$ has a cuspidal support, a function $\overline{\mathcal{S}} : \overline{\mathcal{I}}_0 \to \mathbb{N}_{\geq 0}$ with $\sum_{\sigma \in \overline{\mathcal{I}}_0} \overline{\mathcal{S}}(\sigma) \dim \sigma = n$. For each such $\overline{\mathcal{S}}$, let $\Omega_{\overline{\mathcal{S}}}$ be the full subcategory of $\mathrm{Rep}_{\overline{E}}(\overline{G})$ whose objects are representations all of whose irreducible constituents have cuspidal support $\overline{\mathcal{S}}$.

If σ is a cuspidal representation of $GL_d(k)$ and t is a positive integer, then define the generalised Steinberg representation $\mathrm{St}(\sigma, t)$ to be the unique non-degenerate irreducible representation of $GL_{dt}(k)$ whose cuspidal support is t copies of σ. If $\overline{\mathcal{P}} \in \overline{\mathcal{I}}$ with $\deg \overline{\mathcal{P}} = n$, define a Levi subgroup $\overline{M}_{\overline{\mathcal{P}}}$ of $\overline{G}$ by

$$\overline{M}_{\overline{\mathcal{P}}} = \prod_{\sigma \in \overline{\mathcal{I}}_0, i \in \mathbb{N}} \overline{G}_{\overline{\mathcal{P}}(i) \dim \sigma}.$$

Definition 6.6. Let $\mathrm{St}(\overline{\mathcal{P}})$ be the irreducible representation of $\overline{M}_{\overline{\mathcal{P}}}$ whose tensor factors are the $\mathrm{St}(\sigma, \overline{\mathcal{P}}(\sigma)(i))$ for each (σ, i).

Choose a parabolic subgroup $\overline{Q}$ with Levi factor $\overline{M}_{\overline{\mathcal{P}}}$ and let

$$\pi_{\overline{\mathcal{P}}} = \mathrm{Ind}_{\overline{Q}}^{\overline{G}} \mathrm{St}(\overline{\mathcal{P}}).$$

Definition 6.7. Let $\sigma_{\overline{\mathcal{P}}}$ be the unique irreducible representation contained in $\pi_{\overline{\mathcal{P}}}$ that is not contained in $\pi_{\overline{\mathcal{P}}'}$ for any $\overline{\mathcal{P}}' \succ \overline{\mathcal{P}}$.

Proposition 6.8. *Every irreducible representation of $\overline{G}$ is of the form $\sigma_{\overline{\mathcal{P}}}$ for a unique $\overline{\mathcal{P}}$.*   □

Let R^{GL} be the PSH-algebra defined by taking the dth graded piece R_d^{GL} to be the Grothendieck group of representations of $GL_d(k)$, defining multiplication via parabolic induction, comultiplication via Jacquet restriction, and taking Σ to be the set of isomorphism classes of irreducible representation (see [GR14] §4 for details). The primitive elements in Σ are the cuspidal representations; for each cuspidal representation σ of some $GL_d(k)$ let $R(\sigma)$ be the PSH-subalgebra of R^{GL} spanned by those elements of Σ having cuspidal support some number of copies of σ. Then we have (see the proof of the Proposition in [SZ99] section 4):

Proposition 6.9. *The PSH-algebras $R(\sigma)$ are indecomposable and there is an isomorphism of PSH-algebras*

$$R^{GL} = \bigotimes_{\sigma \in \overline{\mathcal{I}}_0} R(\sigma).$$

For each cuspidal representation σ there is (after rescaling the gradings) a unique isomorphism of PSH-algebras $R(\sigma) \xrightarrow{\sim} R^S$ that takes $\mathrm{St}(\sigma, t)$ to the sign representation of S_t for all t.   □

Corollary 6.10. *If $\overline{\mathcal{P}}, \overline{\mathcal{P}}' \in \overline{\mathcal{I}}$ both have degree n, then the multiplicity*

$$m(\overline{\mathcal{P}}, \overline{\mathcal{P}}') := \dim \mathrm{Hom}_{\overline{G}}(\sigma_{\overline{\mathcal{P}}}, \pi_{\overline{\mathcal{P}}'})$$

is equal to the product of Kostka numbers

$$\prod_{\sigma \in \overline{\mathcal{I}}_0} m(\overline{\mathcal{P}}(\sigma), \overline{\mathcal{P}}'(\sigma)).$$

Proof. First, observe that the bilinear form on R^{GL} is given (on homogeneous elements of the same degree n in the $\mathbb{N}$-span of Σ) by $\dim \mathrm{Hom}_{\overline{G}}(-,-)$. Thus we can read off $m(\overline{\mathcal{P}}, \overline{\mathcal{P}}')$ from the PSH-algebra structure on R^{GL}. By Proposition 6.9, we can reduce to the case where $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}'$ are both supported on the same cuspidal representation σ; let P and P' be $\overline{\mathcal{P}}(\sigma)$ and $\overline{\mathcal{P}}'(\sigma)$ respectively. Then it is easy to see that, under the isomorphism $R(\sigma) \xrightarrow{\sim} R^S$ of Proposition 6.9, $\sigma_{\overline{\mathcal{P}}}$ is taken to σ_P° and $\pi_{\overline{\mathcal{P}}'}$ is taken to $\pi_{P'}^\circ$. The formula follows.   □

6.4. **Simple characters.** We recall a little of the theory of Bushnell and Kutzko (for which see [BK93], [BK98], [BK99]). Let C be an algebraically closed field of characteristic distinct from p (for the case when C has positive characteristic we refer to the works of Vignéras [Vig96], [Vig98] and Mínguez–Sécherre–Stevens [MS14], [SS14]; we will not require much from the positive characteristic theory).

Let V be a vector space over F, let $G = \mathrm{Aut}_F(V)$, and let $A = \mathrm{End}_F(V)$. An $\mathcal{O}_F$-**lattice chain** in V is a sequence $\mathcal{L} = (\Lambda_i)_{i \in \mathbb{Z}}$ of $\mathcal{O}_F$-lattices in V such that $\Lambda_i \supset \Lambda_{i+1}$ for all $i \in \mathbb{Z}$, and such that there exists an integer $e \geq 1$ (the **period** of Λ) with $\Lambda_{i+e} = \mathfrak{p}_F \Lambda_i$ for all $i \in \mathbb{Z}$. The hereditary $\mathcal{O}_F$-orders in A are those orders $\mathfrak{A}$ that arise as the stabiliser of some $\mathcal{O}_F$-lattice chain (which is uniquely determined up to shift by the order). The order $\mathfrak{A}$ is maximal if and only if it stabilises a lattice chain of period $e = 1$. A hereditary order $\mathfrak{A} \subset A$ has a unique two-sided maximal ideal $\mathfrak{P}$; if $\mathfrak{A}$ stabilises Λ then $\mathfrak{P}$ is the set $\{x \in \mathfrak{A} : x\Lambda_i \subset \Lambda_{i+1}$ for all $i \in \mathbb{Z}\}$. We write $U(\mathfrak{A})$ for the group of units in $\mathfrak{A}$ and $U^1(\mathfrak{A}) = 1 + \mathfrak{P}$.

In [BK93] §1.5, the notions of stratum, pure stratum, and simple stratum in A are defined. We will only require simple strata in A of the form $[\mathfrak{A}, m, 0, \beta]$; this means that

- $\mathfrak{A}$ is a hereditary $\mathcal{O}_F$-order in A;
- $m > 0$ is an integer;
- $\beta \in \mathfrak{P}^m \setminus \mathfrak{P}^{1-m}$ is such that $E = F[\beta]$ is a field[6] and $E^\times$ is contained in the normaliser of $U(\mathfrak{A})$;
- $k_0(\beta, \mathfrak{A}(E)) < 0$ where $k_0(\beta, \mathfrak{A}(E))$ is the integer defined in [BK93] §1.4.

If $[\mathfrak{A}, m, 0, \beta]$ is a simple stratum then we may regard V as an E-vector space and write $B = \mathrm{End}_E(V)$. Any lattice chain defining $\mathfrak{A}$ is then an $\mathcal{O}_E$-lattice chain and $\mathfrak{B} := \mathfrak{A} \cap B \subset B$ is its stabiliser; we define the groups $U(\mathfrak{B})$ and $U^1(\mathfrak{B})$ as for $\mathfrak{A}$. To a simple stratum $[\mathfrak{A}, m, 0, \beta]$ we may associate, as in [BK93] §3.1, compact open subgroups $J = J(\beta, \mathfrak{A})$, $J^1 = J^1(\beta, \mathfrak{A})$ and $H = H^1(\beta, \mathfrak{A})$ of $U(\mathfrak{A})$ such that

- J^1 is a normal pro-p subgroup of J;
- H^1 is a normal subgroup of J^1;
- $U(\mathfrak{B}) \subset J$ and $U^1(\mathfrak{B}) \subset J^1$, and the induced map

$$U(\mathfrak{B})/U^1(\mathfrak{B}) \to J/J^1$$

  is an isomorphism.

There is a set $\mathcal{C}(\mathfrak{A}, 0, \beta)$ of **simple characters** of $H^1(\beta, \mathfrak{A})$ (see [BK93] §3.2). If $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ is a simple character, then there is a unique irreducible representation η of $J^1(\beta, \mathfrak{A})$ whose restriction to $H^1(\beta, \mathfrak{A})$ contains θ, and in fact this restriction is a multiple of θ. There is then a distinguished class (the "β-extensions") of extensions κ of η to $J(\beta, \mathfrak{A})$ (see [BK93] §5.2 for char $C = 0$; [Vig96] §4.18 for the general case).

6.5. **Types.** Suppose that char $C = 0$ and Ω is a Bernstein component of $\mathrm{Rep}_C(G)$. A **type** for Ω is a pair (J, λ) where $J \subset G$ is a compact open subgroup and λ is an irreducible representation of J with the property that Ω is equivalent to the category of smooth C-representations of G generated by their λ-isotypic vectors.

Recall that, for H a unimodular locally profinite group, $K \subset H$ a compact open subgroup, and ρ a smooth C-representation of K, then the Hecke algebra $\mathcal{H}(H, K, \rho)$ is defined to be the C-algebra

$$\mathrm{End}_{C[H]}(\text{c-Ind}_K^H(\rho)).$$

If (J, λ) is a type for Ω, then $\mathrm{Hom}_J(\lambda, -)$ is an equivalence of categories between Ω and the category $\mathcal{H}(G, J, \lambda)$-Mod of left $\mathcal{H}(G, J, \lambda)$-modules (see [BK98]).

It is the main result of [BK99] that every Bernstein component of $\mathrm{Rep}_C(G)$ has a type, and there is an explicit construction of these types.

Suppose that Ω is a supercuspidal Bernstein component of $\mathrm{Rep}_C(G)$ (that is, every irreducible object of Ω is supercuspidal). Then, by [BK93] §6 and Theorem 8.4.1, we may construct a type (J, λ) for Ω such that: $J = J(\beta, \mathfrak{A})$ for a simple stratum $[\mathfrak{A}, m, 0, \beta]$ in A in which $\mathfrak{B}$ is a maximal $\mathcal{O}_E$-order, and λ is of the form $\kappa \otimes \nu$ where κ is a β-extension of an irreducible representation η containing a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ and ν is a *cuspidal* representation of $J/J^1 \cong GL_{n/[E:F]}(k_E)$. The integer m is unique and the pair (J, λ) and order $\mathfrak{A}$ are unique up to conjugation in G. A type (J, λ) arising in this way is called a *maximal* type.

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[6]The coefficient field E used in the rest of this paper does not appear in this section.

6.6.   Recall the notions of ps-character and endo-equivalence from [BK99] §4. In the situation of the previous paragraph, the character θ determines a **ps-character** $(\Theta, 0, \beta)$ attached to the simple pair $(0, \beta)$ — this is a function Θ on the set of simple strata $[\mathfrak{A}, m, 0, \beta]$ taking such a stratum to an element $\Theta(\mathfrak{A}) \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. By [BK99] §4.5, the endo-class of this ps-character is determined by Ω. For each endo-class of ps-character we fix a representative $(\Theta, 0, \beta)$. We may and do assume that θ and β in the previous paragraph come from this chosen representative of the endo-class associated to Ω.

We will need to impose a certain compatibility on our choices of β-extensions. Suppose that $(\Theta, 0, \beta)$ is a ps-character attached to the simple pair $(0, \beta)$ and write $E = F[\beta]$. Suppose that E is embedded in $A = \mathrm{End}_F(V)$ so that V is an E-vector space, and let $V_1, \ldots, V_t$ be finite-dimensional E-vector spaces such that $V = \bigoplus_{i=1}^t V_i$. Let M be the corresponding Levi subgroup of G, let Q be the parabolic subgroup with Levi M that stabilises the flag (of F-vector spaces) $0 \subset V_1 \subset V_1 \oplus V_2 \subset \ldots \subset V$, and let U be the unipotent radical of Q. For each i let $A_i = \mathrm{End}_F(V_i)$ and $B_i = \mathrm{End}_E(V_i)$ and let $B = \mathrm{End}_E(V)$. Suppose that, for each i, there is an $\mathcal{O}_E$-lattice $\Lambda_i \subset V_i$ whose stabiliser is $\mathfrak{B}_i$, a maximal hereditary $\mathcal{O}_E$-order in B_i. Let $\mathfrak{A}_i$ be the corresponding hereditary $\mathcal{O}_F$-order in A_i, with associated groups $J_i \supset J_i^1 \supset H_i^1$. Let $\theta_i = \Theta(\mathfrak{A}_i)$ and let η_i be the unique irreducible representation of J_i^1 containing θ_i. Let $\mathfrak{L}$ be the $\mathcal{O}_E$-lattice chain in V whose elements are the lattices

$$\mathfrak{p}_E^a \Lambda_1 \oplus \ldots \oplus \mathfrak{p}_E^a \Lambda_b \oplus \mathfrak{p}_E^{a+1} \Lambda_{b+1} \oplus \ldots \oplus \mathfrak{p}_E^{a+1} \Lambda_t$$

for $a \in \mathbb{Z}$ and $1 \leq b \leq t$ (cf [BK99] §7). Let $\tilde{\mathfrak{B}}$ (resp. $\tilde{\mathfrak{A}}$) be the $\mathcal{O}_E$-order (resp. $\mathcal{O}_F$-order) associated to $\mathfrak{L}$ and let $\mathfrak{B}$ (resp. $\mathfrak{A}$) be the stabiliser in B (resp. A) of a single lattice in $\mathfrak{L}$. Let $\tilde{J} \supset \tilde{J}^1 \supset \tilde{H}^1$ be the groups associated to $\tilde{\mathfrak{A}}$, let $\tilde{\theta} = \Theta(\tilde{\mathfrak{A}})$, and let $\tilde{\eta}$ be the irreducible representation of $\tilde{J}^1$ containing $\tilde{\theta}$. Similarly define $J \supset J^1 \supset H^1$, θ and η to be the objects associated to $\mathfrak{A}$ (and Θ). By [BK93] Theorem 5.2.3, the choice of a β-extension κ of η determines a β-extension $\tilde{\kappa}$ of $\tilde{\eta}$ such that

$$\mathrm{Ind}_{\tilde{J}}^{U(\tilde{\mathfrak{A}})}(\tilde{\kappa}) \cong \mathrm{Ind}_{U(\tilde{\mathfrak{B}})J^1}^{U(\tilde{\mathfrak{A}})}(\kappa|_{U(\tilde{\mathfrak{B}})J^1}).$$

If $\tilde{\kappa}_U$ is the representation of $\tilde{J} \cap M = \prod_{i=1}^t J_i$ on the $\tilde{J} \cap U$-invariants of $\tilde{\kappa}$, then $\tilde{\kappa}_U = \kappa_1 \otimes \ldots \otimes \kappa_t$ for β-extensions κ_i of each η_i. When $\kappa_1, \ldots, \kappa_t$ arise from a single κ in this way, we say that they are **compatible**.

6.7. **Covers.** Types for a general Bernstein component of G are constructed using the formalism of covers. Suppose that $M \subset G$ is a Levi subgroup, that $J \subset G$ is a compact open subgroup, and that ρ is an irreducible smooth representation of J. Write $J_M = J \cap M$ and suppose that $\rho_M = \rho|_{J_M}$ is irreducible. The notion of (J, ρ) being a G**-cover** of (J_M, ρ_M) is defined in [BK98] Definition 8.1.

By [BK98] Theorem 7.2, if (J, ρ) is a G-cover of (J_M, ρ_M), then for each parabolic subgroup Q of G with Levi factor M, there is an injective Hecke algebra homomorphism

$$j_Q : \mathcal{H}(M, J_M, \rho_M) \to \mathcal{H}(G, J, \rho).$$

Moreover, if every element of G intertwining ρ lies in M, then j_Q is an isomorphism, by Theorem 7.2 and the remark following Corollary 7.7 of [BK98].

If $[M, \pi]$ is an inertial equivalence class of supercuspidal pair corresponding to a Bernstein component Ω of $\mathrm{Rep}_C(G)$, then let (J_M, λ_M) be a maximal type for the

supercuspidal Bernstein component Ω_M of $\mathrm{Rep}_C(M)$ containing π. By the results of [BK99], there is a G-cover (J, λ) of (J_M, λ_M). The pair (J, λ) is then a type for Ω. For every parabolic subgroup $Q \subset G$ with Levi subgroup M, the diagram

(5)
$$
\begin{array}{ccc}
\Omega_M & \xrightarrow{\mathrm{Ind}_Q^G(-)} & \Omega \\
{\scriptstyle \mathrm{Hom}_{J_M}(\lambda_M, -)} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Hom}_J(\lambda, -)} \\
\mathcal{H}(M, J_M, \lambda_M)\text{-Mod} & \xrightarrow{\ j_Q\ } & \mathcal{H}(G, J, \lambda)\text{-Mod}
\end{array}
$$

commutes (by [BK98] Corollary 8.4).

6.8. **SZ-data.** We return to the case of arbitrary C with characteristic distinct from p.

Definition 6.11. An SZ-datum over C is a set

$$\{(E_i, \beta_i, V_i, \mathfrak{B}_i, \lambda_i)\}_{i=1}^r$$

where r is a positive integer and, for each $i = 1, \ldots, r$, we have:

- E_i/F is a finite extension generated by an element $\beta_i \in E_i$;
- V_i is an E_i-vector space of finite dimension N_i;
- $\mathfrak{B}_i \subset \mathrm{End}_{E_i}(V_i)$ is a maximal hereditary $\mathcal{O}_{E_i}$-order and $\mathfrak{A}_i$ is the associated $\mathcal{O}_F$-order in $A_i := \mathrm{End}_F(V_i)$;
- if $m_i = -v_{E_i}(\beta_i)$, then $[\mathfrak{A}_i, m_i, 0, \beta_i]$ is a simple stratum and λ_i is a C-representation of $J_i = J(\beta_i, \mathfrak{A}_i)$ of the form $\kappa_i \otimes \nu_i$. Here κ_i is a β_i-extension of the representation η_i of $J_i^1 = J^1(\beta_i, \mathfrak{A}_i)$ containing some simple character $\theta_i \in \mathcal{C}(\mathfrak{A}_i, 0, \beta_i)$ of $H_i^1 = H^1(\beta_i, \mathfrak{A}_i)$, and ν_i is an irreducible representation of $U(\mathfrak{B}_i)/U^1(\mathfrak{B}_i) \cong GL_{N_i}(k_{E_i})$ over C;
- no two of the θ_i are endo-equivalent.

Suppose that $\mathfrak{S} = \{(E_i, \beta_i, V_i, \mathfrak{B}_i, \lambda_i)\}_{i=1}^r$ is an SZ-datum, and adopt all of the above notation (including the implied choices of β_i-extensions κ_i).

Proposition 6.12. *The representations λ_i are irreducible.*

Proof. Suppose that some λ_i is reducible. Since $(\kappa_i \otimes \nu_i)|_{H_i^1}$ is a multiple of θ_i, we must have that any irreducible subrepresentation ρ of $\lambda_i = \kappa_i \otimes \nu_i$ contains θ_i when restricted to H_i^1. Therefore, by [Vig96], 4.22 Lemme, ρ must also be of the form $\kappa_i \otimes \nu_i'$ for an irreducible representation ν_i' of J_i/J_i^1. But now by [Vig96], 4.22 "Entrelacement", we have

$$\mathrm{Hom}_{J_i}(\kappa_i \otimes \nu_i', \kappa_i \otimes \nu_i) = \mathrm{Hom}_{J_i/J_i^1}(\nu_i', \nu_i)$$

and so we must have $\nu_i' = \nu_i$, as ν_i is irreducible. Therefore $\rho = \kappa_i \otimes \nu_i = \lambda_i$ as required. $\square$

Let $V = \bigoplus_{i=1}^r V_i$ (an F-vector space), $A = \mathrm{End}_F(V)$ and $G = \mathrm{Aut}_F(V)$. The Levi subgroup $M = \prod_{i=1}^r \mathrm{Aut}_F(A_i) \subset G$ has compact open subgroups $J_M^1 \lhd J_M$, where $J_M^1 = \prod_{i=1}^r J_i^1$ and similarly for J. Let $\eta_M = \bigotimes_{i=1}^r \eta_i$ (a representation of J_M^1) and similarly define the representations κ_M and λ_M of J_M. Then η_M and κ_M are clearly irreducible, and λ_M is irreducible by the above proposition.

Since no two of the θ_i are endo-equivalent, the constructions of [BK99] §8 (see also [MS14] §§2.9-10) yield compact open subgroups J and J^1 of G and representations

η of J^1, κ of J and λ of J such that (J^1, η) (resp. (J, κ), resp. (J, λ)) is a G-cover of (J_M^1, η_M) (resp. (J_M, κ), resp. (J_M, λ_M)), and $J/J^1 = J_M/J_M^1$ with $\lambda = \kappa \otimes (\bigotimes_{i=1}^r \nu_i)$ under this identification.

Remark 6.13. If char $C \neq 0$, then to see that these are G-covers we must modify the proof of [BK99] Corollary 6.6 as explained in the proof of [MS14] Proposition 2.28. However, here we only need the case char $C = 0$.

Let K be a maximal compact subgroup of G such that $J_M \subset K \cap M$ (and so $K \cap \prod_{i=1}^r B_i = \prod_{i=1}^r U(\mathfrak{B}_i)$).

Proposition 6.14. *Every element of K that intertwines η lies in J.*

Proof. By [MS14] Proposition 2.31, the G-intertwining of η is $J(\prod_{i=1}^r B_i^\times)J$, and so the K-intertwining of η is

$$J \left(\prod_{i=1}^r B_i^\times \right) J \cap K = J. \qquad \square$$

Definition 6.15. Let $\mathfrak{S}$ be an SZ-datum over C and let J, K and λ be as above. Then:

$$\sigma(\mathfrak{S}) = \mathrm{Ind}_J^K(\lambda).$$

Theorem 6.16. *The representation $\sigma(\mathfrak{S})$ is irreducible.*

Proof. We first show that

$$\dim \mathrm{Hom}_{J^1}(\eta, \mathrm{Ind}_J^K \lambda) = \dim(\nu).$$

By Proposition 6.14, for $g \notin J$ we have $\mathrm{Hom}_{J^1 \cap J^g}(\eta, \lambda^g) = 0$. Therefore by Mackey's formula,

$$\dim \mathrm{Hom}_{J^1}(\eta, \mathrm{Ind}_J^K \lambda) = \dim \mathrm{Hom}_{J^1}(\eta, \lambda)$$
$$= \dim(\nu)$$

Now suppose that $\mathrm{Ind}_J^K \lambda$ is reducible, with

$$0 \subsetneq W \subsetneq \mathrm{Ind}_J^K \lambda$$

a K-submodule and W' the quotient. We may write $\mathrm{Res}_{J^1}^K \mathrm{Ind}_J^K \lambda = W \oplus W'$, since J^1 is pro-p. Now, by Frobenius reciprocity we have that

$$\dim \mathrm{Hom}_J(W, \lambda) \geq 1.$$

Since λ is irreducible and $\lambda|_{J^1} = \dim(\nu) \cdot \eta$ this shows that

$$\dim \mathrm{Hom}_{J^1}(\eta, W) = \dim \mathrm{Hom}_{J^1}(W, \eta) \geq \dim(\nu).$$

But the same argument applies to W', so that

$$\dim \mathrm{Hom}_{J^1}(\eta, \mathrm{Ind}_J^K(\lambda)) \geq 2 \dim \nu > \dim \nu,$$

a contradiction! $\qquad \square$

6.9. K-types. Now take $C = \overline{E}$. Let $\mathcal{P} \in \mathcal{I}$, let $\mathrm{scs}(\mathcal{P}) = \mathcal{S} : \mathcal{I}_0 \to \mathbb{Z}_{\geq 0}$ be as in section 3, and let $\Omega = \Omega_{\mathcal{S}}$ be the associated Bernstein component of $\mathrm{Rep}_C(G)$. Let $n = \sum_{\tau_0 \in \overline{\mathcal{I}}_0} \dim \tau_0 \mathcal{S}(\tau_0)$ and let $G = \mathrm{Aut}_F(V)$ for an n-dimensional F-vector space V. Let (M^0, π) be a supercuspidal pair in the inertial equivalence class associated to Ω. Write $M^0 = \prod_{i=1}^t M_i^0$ with each M_i^0 the stabiliser of some n_i-dimensional subspace V_i^0 of V, write $\pi = \bigotimes_{i=1}^t \pi_i$, and let Ω_i be the supercuspidal Bernstein component of $\mathrm{Rep}_C(M_i^0)$ containing π_i. For each Ω_i there is an associated endo-class of ps-character, for which we have chosen a representative $\Theta_i^0 = (\Theta_i^0, 0, \beta_i^0)$. Construct a Levi subgroup $M = \prod_{i=1}^r M_i$ with $M^0 \subset M \subset G$ by requiring that M_j^0 and M_k^0 are both contained in some M_i if and only if $\Theta_j^0 = \Theta_k^0$; in this case we write

$$(\Theta_i, 0, \beta_i) = (\Theta_j^0, 0, \beta_j^0) = (\Theta_k^0, 0, \beta_k^0)$$

for the common value. Let $V = \bigoplus_{i=1}^r V_i = \bigoplus_{i=1}^t V_i^0$ be the decompositions of V corresponding to M and M^0 respectively, so that the second is strictly finer than the first.

Suppose first that $r = 1$ (the **homogeneous** case). Then write $(\Theta, 0, \beta)$ for the common value of $(\Theta_i^0, 0, \beta_i^0)$, and $E = F[\beta]$. For $0 \leq i \leq t$, there is a maximal simple type (J_i^0, λ_i^0) for Ω_i such that $J_i^0 = J(\beta, \mathfrak{A}_i^0)$ for a simple stratum $[\mathfrak{A}_i^0, m, 0, \beta]$, and λ_i^0 contains $\theta_i^0 := \Theta(\mathfrak{A}_i^0, 0, \beta)$. We are in the situation of section 6.6, and adopt the notation there (adorning it with a superscript '0' where appropriate). In particular we have compact open subgroups $J^1 \subset J$ of G and a representation η of J^1 containing the simple character $\Theta(\mathfrak{A}, 0, \beta)$, where $\mathfrak{A}$ is a hereditary $\mathcal{O}_F$-order in A and $\mathfrak{A} \cap B = \mathfrak{B}$ is a maximal hereditary $\mathcal{O}_E$-order. We choose *compatible* β-extensions κ_i^0 of η_i^0 coming from a β-extension κ of η, and decompose each λ_i^0 as $\kappa_i^0 \otimes \nu_i^0$, where ν_i^0 is a cuspidal representation of $J_i^0/J_i^{1,0} = U(\mathfrak{B}_i^0)/U^1(\mathfrak{B}_i^0)$. Choosing an $\mathcal{O}_E$-basis of each $\mathfrak{B}_i^0$, we identify $U(\mathfrak{B}_i^0)/U^1(\mathfrak{B}_i^0)$ with $GL_{n_i^0/[E:F]}(k_E)$ for an integer n_i^0 and $J/J^1 = U(\mathfrak{B})/U^1(\mathfrak{B})$ with $GL_{n/[E:F]}(k_E)$. So we may view each ν_i^0 as an element of $\overline{\mathcal{I}}_0$, and define an element $\overline{\mathcal{P}} \in \overline{\mathcal{I}}$ by $\overline{\mathcal{P}}(\nu_i^0) = \mathcal{P}(\tau_i)$, where $\tau_i \in \mathcal{I}_0$ corresponds to Ω_i. Then write $\nu = \pi_{\overline{\mathcal{P}}}$, a representation of $GL_{n/[E:F]}(k_E)$, and regard it as a representation of J/J^1.

Definition 6.17. In this homogeneous case, we define an SZ-datum $\mathfrak{S}_{\mathcal{P}}$ by

$$\mathfrak{S}_{\mathcal{P}} = \{(E, \beta, V, \mathfrak{B}, \kappa \otimes \nu)\}.$$

In the general case, for $1 \leq i \leq r$ let

$$\{(E_i, \beta_i, V_i, \mathfrak{B}_i, \kappa_i \otimes \nu_i)\}$$

be the SZ-datum for M_i given by the construction in the homogeneous case, and set

$$\mathfrak{S}_{\mathcal{P}} = \{(E_i, \beta_i, V_i, \mathfrak{B}_i, \kappa_i \otimes \nu_i)\}_{i=1}^r.$$

If $\tau = \tau_{\mathcal{P}}$, we write $\sigma(\tau) = \sigma(\mathfrak{S}_{\mathcal{P}})$.

6.10. We show that the representations $\sigma(\tau)$ satisfy the conclusion of Theorem 3.7. Continue with the notation of section 6.9, and suppose that $r = 1$. Let $M^2 \supset M^0$ be a Levi subgroup of a parabolic subgroup $Q^2 \subset G$, let $\mathfrak{B}^2 = \mathfrak{B} \cap M^2$, and let $J^{1,2} \subset J^2$, η^2 and κ^2 be the subgroups of M^2 and their representations obtained from Θ. We require that κ^2 is compatible with κ.

Write $\overline{G} = J/J^1 = U(\mathfrak{B})/U^1(\mathfrak{B})$, $\overline{M}^2 = (M^2 \cap U(\mathfrak{B}))J^1/J^1$, and $\overline{Q}^2 = (Q^2 \cap U(\mathfrak{B}))J^1/J^1$. Then $\overline{Q}^2$ is a parabolic subgroup of $\overline{G}$ with Levi $\overline{M}^2$.

Proposition 6.18. *The following diagram commutes:*

(6)
$$
\begin{array}{ccc}
\mathrm{Rep}_C(M^2) & \xrightarrow{\ \mathrm{Ind}_{Q^2}^G(-)\ } & \mathrm{Rep}_C(G) \\
{\scriptstyle \mathrm{Hom}_{J^{1,2}}(\kappa^2,-)}\Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Hom}_{J^1}(\kappa,-)} \\
\mathrm{Rep}_C(\overline{M}^{\,2}) & \xrightarrow[\ \mathrm{Ind}_{\overline{Q}^2}^{\overline{G}}(-)\ ]{} & \mathrm{Rep}_C(\overline{G}).
\end{array}
$$

Proof. This may be proved in the same way as [SZ99] Proposition 7; we omit the details. See also [SS14] Proposition 5.6. $\square$

Corollary 6.19. *Suppose that M^2 is as above and further suppose that (M^2, π^2) is a discrete pair in the inertial equivalence class associated to some $\mathcal{P}' \in \mathcal{I}$; let $\overline{\mathcal{P}}' : \overline{\mathcal{I}}_0 \to \mathrm{Part}$ correspond to $\mathcal{P}'$. Then*

$$
\mathrm{Hom}_{J^1}(\kappa, \mathrm{Ind}_{Q^2}^G(\pi^2)) = \pi_{\overline{\mathcal{P}}'},
$$

as representations of $\overline{G}$.

Proof. By Proposition 6.18,

$$
\mathrm{Hom}_{J^1}(\kappa, \mathrm{Ind}_{Q^2}^G(\pi^2)) = \mathrm{Ind}_{\overline{Q}^2}^{\overline{G}}(\mathrm{Hom}_{J^{1,2}}(\kappa^2, \pi^2)).
$$

By [SZ99] Proposition 5.6,

$$
\mathrm{Hom}_{J^{1,2}}(\kappa^2, \pi^2) = \mathrm{St}(\overline{\mathcal{P}}').
$$

Therefore:

$$
\mathrm{Ind}_{\overline{Q}^2}^{\overline{G}}(\mathrm{Hom}_{J^{1,2}}(\kappa^2, \pi^2)) = \mathrm{Ind}_{\overline{Q}^2}^{\overline{G}}(\mathrm{St}(\overline{\mathcal{P}}'))
$$
$$
= \pi_{\overline{\mathcal{P}}'}. \qquad\qquad \square
$$

Now suppose that $r > 1$, so that $M \subset G$ is a proper Levi subgroup. Let $J_M = \prod_{i=1}^r J_i$, $J_M^1 = \prod_{i=1}^r J_i^1$, and $\eta_M = \bigotimes_{i=1}^r \eta$. Then as in section 6.8 there is a G-cover (J^1, η) of (J_M^1, η_M). We have a canonical isomorphism $J_M/J_M^1 = J/J^1$ induced by the inclusion $J_M \hookrightarrow J$. For each parabolic subgroup Q of G with Levi M, there is an isomorphism

$$
j_Q : \mathcal{H}(M, J_M^1, \eta_M) \xrightarrow{\sim} \mathcal{H}(G, J^1, \eta)
$$

such that the diagram

(7)
$$
\begin{array}{ccc}
\mathrm{Rep}_C(M) & \xrightarrow{\ \mathrm{Ind}_Q^G(-)\ } & \mathrm{Rep}_C(G) \\
{\scriptstyle \mathrm{Hom}_{J_M^1}(\eta_M,-)}\Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Hom}_{J^1}(\eta,-)} \\
j_Q : \mathcal{H}(M, J_M, \eta_M)\text{-Mod} & \xrightarrow{\ \sim\ } & \mathcal{H}(G, J, \eta)\text{-Mod}
\end{array}
$$

commutes, by the discussion of section 6.7 and the intertwining bound of [MS14] Proposition 2.31. Then, writing $K_M = K \cap M$, j_Q induces an isomorphism

$$
\mathcal{H}(K_M, J_M^1, \eta_M) \xrightarrow{\sim} \mathcal{H}(K, J^1, \eta).
$$

But, by Proposition 6.14, we have

$$
\mathcal{H}(K_M, J_M^1, \eta_M) = \mathcal{H}(J_M, J_M^1, \eta_M)
$$

and choosing κ_M identifies this with $C[J_M/J_M^1]$. Similarly, choosing κ identifies $\mathcal{H}(K, J^1, \eta)$ with $C[J/J^1]$. As j_Q is support-preserving, if we choose κ such that

$\kappa|_{J_M} = \kappa_M$ then the isomorphism j_Q agrees with the identification $C[J_M/J_M^1] = C[J/J^1]$. Therefore, when ν is a representation of $J/J^1 = J_M/J_M^1$, the isomorphism j_Q takes $\mathrm{Ind}_{J_M}^{K_M}(\kappa_M \otimes \nu)$ to $\mathrm{Ind}_J^K(\kappa \otimes \nu)$. So we have shown that, for every smooth representation π_M of M, we have:

$$(8) \qquad \mathrm{Hom}_K(\mathrm{Ind}_J^G(\kappa \otimes \nu), \mathrm{Ind}_Q^G(\pi_M)) = \mathrm{Hom}_{K_M}(\mathrm{Ind}_{J_M}^{K_M}(\kappa_M \otimes \nu_M), \pi_M).$$

Theorem 6.20. *Let* $\mathcal{P}' \in \mathcal{I}$ *with* $\deg \mathcal{P}' = n$, *let* (M', π') *be any discrete pair in the inertial equivalence class associated to* $\mathcal{P}'$, *and let* $Q' \subset G$ *be any parabolic subgroup with Levi subgroup* M'. *Then*

$$\dim \mathrm{Hom}_K(\sigma(\mathfrak{S}_{\mathcal{P}}), \mathrm{Ind}_{Q'}^G(\pi')) = \prod_{\tau_0 \in \mathcal{I}_0} m(\mathcal{P}(\tau_0), \mathcal{P}'(\tau_0)).$$

Proof. We can assume that $\mathrm{scs}(\mathcal{P}') = \mathfrak{S} = \mathrm{scs}(\mathcal{P})$; otherwise both sides are zero — the left hand side because $\sigma(\mathfrak{S}_{\mathcal{P}})$ contains a type for the Bernstein component Ω corresponding to $\mathcal{S}$. Therefore we can assume that $M_0 \subset M' \subset M$. Using the commutative diagram (7), we may reduce to the case in which $\mathcal{P}$ and $\mathcal{P}'$ are homogeneous. But now the result follows from Corollary 6.19 and Corollary 6.10. $\square$

Corollary 6.21. *Let* $\mathcal{P}' \in \mathcal{I}$ *and let* (M', π') *be a discrete pair in the inertial equivalence class associated to* $\mathcal{P}'$. *Let* $\pi = L(M', \pi')$ *be the irreducible admissible representation defined in section 3.5, so that* $\mathrm{rec}_l(\pi)|_{I_F} \cong \tau_{\mathcal{P}'}$. *Then:*

(1) *if* $\pi|_K$ *contains* $\sigma(\mathfrak{S}_{\mathcal{P}})$, *then* $\mathcal{P}' \preceq \mathcal{P}$;
(2) *if* $\mathcal{P}' = \mathcal{P}$, *then* $\pi|_K$ *contains* $\sigma(\mathfrak{S}_{\mathcal{P}})$ *with multiplicity one;*
(3) *if* $\mathcal{P}' \preceq \mathcal{P}$ *and* π *is* generic, *then* $\pi|_K$ *contains* $\sigma(\mathfrak{S}_{\mathcal{P}})$ *with multiplicity one.*

Proof. This is proved in the same way as [SZ99] Proposition 5.10. By Theorem 6.20, if $Q' \subset G$ is a parabolic subgroup with Levi M', then $\sigma(\mathfrak{S}_{\mathcal{P}})$ is contained in $\mathrm{Ind}_{Q'}^G(\pi')$ if and only if $\mathcal{P}' \preceq \mathcal{P}$. Therefore if $L(M', \pi')$ contains $\sigma(\mathfrak{S}_{\mathcal{P}})$, then $\mathcal{P}' \preceq \mathcal{P}$, proving part 1. If $L(M', \pi')$ is generic, then it is equal to $\mathrm{Ind}_{Q'}^G(\pi')$ for any Q', proving part 3. Finally, suppose $\mathcal{P}' = \mathcal{P}$. By the theorem, $\sigma(\mathfrak{S}_{\mathcal{P}})$ occurs in $\mathrm{Ind}_{Q'}^G(\pi')$ with multiplicity one; in other words, exactly one constituent of $\mathrm{Ind}_{Q'}^G(\pi')$ contains $\sigma(\mathfrak{S}_{\mathcal{P}})$, and it does so with multiplicity one. But every constituent of $\mathrm{Ind}_{Q'}^G(\pi')$ other than $L(M', \pi')$ is equal to $L(M'', \pi'')$ for some discrete pair (M'', π'') in the inertial equivalence class associated to $\mathcal{P}''$ for some $\mathcal{P}'' \succ \mathcal{P}$ (see [SZ99] §2 Lemma), and so by part 1 does not contain $\sigma(\mathcal{P})$. Hence $\sigma(\mathcal{P})$ is contained in $L(M', \pi')$ with multiplicity one, as required. $\square$

Corollary 6.22. *Let* $\mathcal{P}$ *and* $\mathcal{P}'$ *be elements of* $\mathcal{I}$. *Then*

$$m(\sigma(\tau_{\mathcal{P}}), \tau_{\mathcal{P}'}) = \prod_{\tau_0 \in \mathcal{I}_0} m(\mathcal{P}(\tau_0), \mathcal{P}'(\tau_0)).$$

Proof. This follows from Theorem 6.20 together with that fact that any *generic* irreducible admissible representation π of $GL_n(F)$ is the *irreducible* induction of a discrete series representation of a Levi subgroup. $\square$

6.11. **Reduction modulo l.** Let $\mathfrak{S} = \{(E_i, \beta_i, V_i, \mathfrak{B}_i, \lambda_i)\}_{i=1}^r$ be an SZ-datum over $\overline{E}$. Decompose each λ_i as $\kappa_i \otimes \nu_i$ for irreducible representations ν_i of J_i/J_i^1. Suppose that

$$\overline{\nu_i}^{ss} = \bigoplus_{j \in S_i} \mu_{ij} \nu_{ij}$$

where S_i is some finite indexing set, ν_{ij} are distinct irreducible representations of J_i/J_i^1 over $\overline{\mathbb{F}}$ and $\mu_{ij} \in \mathbb{N}$. Note that each $\overline{\eta_i}$, and hence $\overline{\kappa_i}$, is irreducible. For $\boldsymbol{j} = (j_1, \ldots, j_r) \in S_1 \times \ldots \times S_r$, define an SZ-datum $\mathfrak{S}_{\boldsymbol{j}}$ over $\overline{\mathbb{F}}$ by

$$\mathfrak{S}_{\boldsymbol{j}} = \{(E_i, \beta_i, V_i, \mathfrak{B}_i, \overline{\kappa_i} \otimes \nu_{ij_i}\}_{i=1}^r$$

and an integer $\mu_{\boldsymbol{j}} = \prod_{i=1}^r \mu_{ij_i}$. Then we have:

Theorem 6.23. *The semisimplified mod l reduction of $\sigma(\mathfrak{S})$ is*

$$\bigoplus_{\boldsymbol{j} \in S_1 \times \ldots \times S_r} \mu_{\boldsymbol{j}} \sigma(\mathfrak{S}_{\boldsymbol{j}}).$$

Proof. This follows immediately from Theorem 6.16. $\qquad\square$

7. Towards a local proof

In this section, we give a proof of Conjecture 4.5 in the case that $q \equiv 1 \mod l$ and $l > n$ (we say that l is **quasi-banal**), $\overline{\rho}|_{\tilde{P}_F}$ is trivial, and $R_E(GL_n(\mathcal{O}_F))$ and $R_{\mathbb{F}}(GL_n(\mathcal{O}_F))$ are replaced by subgroups generated by certain representations inflated from $GL_n(k_F)$. The strategy of proof is to first show that it suffices to prove Conjecture 4.5 for a single $\overline{\rho}$ on each irreducible component of $\mathcal{M}(n, q)_{\mathbb{F}}$ such that $\overline{\rho}$ is on no other irreducible components. But for good choices of $\overline{\rho}$, we may explicitly determine the rings $R^{\square}(\overline{\rho}, \tau)$ for all inertial types τ. As we also have a very good understanding of the mod l representation theory of $GL_n(k_F)$ under our assumptions on l, Conjecture 4.5 reduces to a combinatorial identity, which we verify.

7.1. **Reduction to finite type.** Let $\mathfrak{X}$ be the affine scheme $\mathcal{M}(n, q)_{\mathcal{O}}$ from section 2. We suppose that $\mathcal{O}$ contains all of the $(q^{n!}-1)$th roots of unity, so that every irreducible component of $\mathfrak{X}_{\mathbb{F}}$ or $\mathfrak{X}_E$ is geometrically irreducible. Once we have fixed generators σ and ϕ for G_F/P_F as usual, then there is a natural bijection between $\mathfrak{X}(\overline{\mathbb{F}})$ and the set of continuous homomorphisms $\overline{\rho} : G_F \to GL_n(\overline{\mathbb{F}})$ with kernel containing P_F. If x is a closed point of $\mathfrak{X}$ corresponding to such a homomorphism $\overline{\rho}_x$, and we suppose that the residue field of $\mathfrak{X}$ at x is $\mathbb{F}$, then there is a natural isomorphism

$$\mathcal{O}_{\mathfrak{X},x}^{\wedge} = R^{\square}(\overline{\rho}_x).$$

From the map

$$i : \operatorname{Spec} R^{\square}(\overline{\rho}_x) \to \mathfrak{X}$$

we get a pullback

$$i^* : \mathcal{Z}(\mathfrak{X}) \to \mathcal{Z}(R^{\square}(\overline{\rho}_x))$$

as in section 2.3. Similarly, writing $\overline{\mathfrak{X}} = \mathfrak{X} \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathbb{F}$, we have a map

$$i^* : \mathcal{Z}(\overline{\mathfrak{X}}) \to \mathcal{Z}(R^{\square}(\overline{\rho}_x) \otimes_{\mathcal{O}} \mathbb{F}),$$

and the diagram

$$(9) \qquad \begin{array}{ccc} \mathcal{Z}(\mathfrak{X}) & \xrightarrow{i^*} & \mathcal{Z}(R^{\square}(\overline{\rho}_x)) \\ {\scriptstyle\text{red}}\downarrow & & {\scriptstyle\text{red}}\downarrow \\ \mathcal{Z}(\overline{\mathfrak{X}}) & \xrightarrow{i^*} & \mathcal{Z}(\overline{R}^{\square}(\overline{\rho}_x)) \end{array}$$

commutes, by Lemma 2.9.

There is a unique map

$$\text{cyc}_{\text{ft}} : R_E(GL_n(\mathcal{O}_F)) \to \mathcal{Z}(\mathfrak{X})$$

such that for each $x \in \overline{\mathfrak{X}}(\mathbb{F})$ the map $\text{cyc} : R_E(GL_n(\mathcal{O}_F)) \to \mathcal{Z}(R^\square(\overline{\rho}_x))$, which to avoid ambiguity we will call cyc_x, is equal to the composition $i^* \circ \text{cyc}_{\text{ft}}$.

Let BM_{ft} be the statement that there exists a map $\overline{\text{cyc}}_{\text{ft}}$ (necessarily unique) making the diagram

(10)
$$\begin{array}{ccc} R_E(GL_n(\mathcal{O}_F)) & \xrightarrow{\text{cyc}_{\text{ft}}} & \mathcal{Z}(\mathfrak{X}) \\ {\scriptstyle \text{red}} \downarrow & & {\scriptstyle \text{red}} \downarrow \\ R_{\mathbb{F}}(GL_n(\mathcal{O}_F)) & \xrightarrow{\overline{\text{cyc}}_{\text{ft}}} & \mathcal{Z}(\overline{\mathfrak{X}}). \end{array}$$

commute, and for $x \in \mathfrak{X}(\mathbb{F})$ let BM_x be the statement that Conjecture 4.5 holds for $\overline{\rho} = \overline{\rho}_x$. Then we have:

Proposition 7.1.   (1) *If* BM_{ft} *is true, so is* BM_x *for all* $x \in \mathfrak{X}(\mathbb{F})$.
     (2) *Suppose that* $S \subset \mathfrak{X}(\mathbb{F})$ *has the property that, for every irreducible component* $\mathfrak{Z}$ *of* $\overline{\mathfrak{X}}$, *there is an* $x \in S$ *such that* x *lies on* $\mathfrak{Z}$ *and on no other irreducible component of* $\overline{\mathfrak{X}}$. *If* BM_x *is true for all* $x \in S$, *then* BM_{ft} *is true.*

Proof. For the first part, given the existence of a map $\overline{\text{cyc}}_{\text{ft}}$ we define $\overline{\text{cyc}}_x$ to be the composition of $\overline{\text{cyc}}_{\text{ft}}$ with i^*. Then BM_x follows from the commutativity of diagrams (9) and (10).

For the second part we simply need to observe that, under the given assumptions on S, the map

$$i^* : \mathcal{Z}(\overline{\mathfrak{X}}) \to \prod_{x \in S} \mathcal{Z}(\overline{R}^\square(\overline{\rho}_x))$$

is injective. $\square$

Let $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$ be a representation $(\overline{M}, \overline{\rho})$ of G_F with a basis $(\overline{e}_i)_i$. Let $\overline{M} = \overline{M}_1 \oplus \ldots \oplus \overline{M}_r$ for the decomposition of $\overline{M}$ into generalised eigenspaces for $\overline{\rho}(\phi)$, with $\overline{M}_i$ having generalised eigenvalue $\alpha_i \in \mathbb{F}$ and dimension n_i.

Definition 7.2. Say that $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$ is **standard** if each $\overline{e}_i$ lies in some $\overline{M}_j$.

Let A be an object of $\mathcal{C}_{\mathcal{O}}$ and let $(M, \rho, (e_i)_i)$ be a lift of $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$. Say that $(M, \rho, (e_i)_i)$ is **standard** if we may write $M = M_1 \oplus \ldots \oplus M_r$ with each M_i being a $\rho(\phi)$-stable lift of $\overline{M}_i$ and, whenever $\overline{e}_i \in \overline{M}_j$ for some i, j, we have $e_i \in M_j$.

The property of being standard only depends on the equivalence class of $(M, \rho, (e_i)_i)$, and so we can talk of homomorphisms $\rho : G_F \to GL_n(A)$ being standard.

Let $R^{\text{std}}(\overline{\rho})$ be the maximal quotient of $R^\square(\overline{\rho})$ on which $\rho^\square$ is standard.

Thus we are requiring that $\overline{\rho}(\phi)$ is block diagonal with each block having a single generalised eigenvalue and different blocks having different eigenvalues, and that $\rho(\phi)$ is block diagonal with blocks lifting those of $\overline{\rho}(\phi)$. It is clear that, given $(\overline{M}, \overline{\rho})$, we may choose a basis $\overline{e}_i$ such that $(\overline{M}, \overline{\rho}, (\overline{e}_i))$ is standard.

Lemma 7.3. *Let* $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$ *be standard. Then there is an injective morphism*

$$R^{\text{std}}(\overline{\rho}) \to R^\square(\overline{\rho})$$

in $\mathcal{C}_{\mathcal{O}}^\wedge$ *making* $R^\square(\overline{\rho})$ *formally smooth over* $R^{\text{std}}(\overline{\rho})$.

Proof. Adopt the notation of Definition 7.2. Let $\overline{P}_i(X) = (X - \alpha_i)^{n_i}$ for $i = 1, \ldots, r$, so that the characteristic polynomial of $\overline{\rho}(\phi)$ is

$$\overline{P}(X) = \prod_{i=1}^r \overline{P}_i(X).$$

If $(M, \rho, (e_i)_i)$ is a lift of $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$ to $A \in \mathcal{C}_{\mathcal{O}}$, we will functorially produce a new basis $(f_i)_i$ such that $(M, \rho, (f_i)_i)$ is standard. Let $P(X)$ be the characteristic polynomial of $\rho(\phi)$. By Hensel's lemma, there is a factorisation

$$P(X) = \prod_{i=1}^r P_i(X)$$

with $P_i(X) \in A[X]$ such that the image of $P_i(X)$ in $\mathbb{F}[X]$ is $\overline{P}_i(X)$ for each i. Let

$$Q_i(X) = \frac{P(X)}{P_i(X)} = \prod_{j \neq i} P_j(X)$$

for $1 \leq i \leq r$. Writing $M_i = Q_i(\rho(\phi))M$, we have

$$M = \bigoplus_{i=1}^r M_i.$$

Then the isomorphism $M \otimes \mathbb{F} \xrightarrow{\sim} \overline{M}$ takes M_i to $\ker(\overline{P}_i) = \overline{M}_i$ and each M_i is a $\rho(\phi)$-stable submodule of M.

Now, each e_i may be written uniquely as $e_i^{(1)} + \ldots + e_i^{(r)}$ with $e_i^{(j)} \in M_j$ for each j; we take $f_i = e_i^{(i)}$. Then $(M, \rho, (f_i)_i)$ is a standard lift of $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$.

We have therefore defined a map $R^{\mathrm{std}}(\overline{\rho}) \to R^{\square}(\overline{\rho})$ which is easily seen to be injective and formally smooth. $\qquad\square$

7.2. **Representation theory.** From now until the end of section 7, we suppose that l is quasi-banal — that is, that $l > n$ and $q \equiv 1 \bmod l$. Let $a = v_l(q-1)$ and let μ_{l^a} be the group of l^ath roots of unity in $\mathcal{O}$. Let $T \subset B \subset GL_n(k_F)$ be the standard maximal torus and Borel subgroup, let U be the unipotent radical of B, and let B_1 be the maximal subgroup of B of order coprime to l, so that $B/B_1 \cong (\mathbb{Z}/l^a\mathbb{Z})^n$. Let $R_E^1(GL_n(k_F)) \subset R_E(GL_n(k_F))$ and $R_{\mathbb{F}}^1(GL_n(k_F)) \subset R_{\mathbb{F}}(GL_n(k_F))$ be the subgroups generated by those irreducible representations having a B_1-fixed vector.

Recall the notation $\overline{\mathcal{I}}$, $\overline{\mathcal{I}}_0$, $\pi_{\overline{\mathcal{P}}}$, $\sigma_{\overline{\mathcal{P}}}$ from section 6.3. If χ is a character of $k_F^\times$ with values in μ_{l^a}, then χ is an element of $\overline{\mathcal{I}}_0$ of degree one. Let $\overline{\mathcal{I}}_1$ be the set of functions $\overline{\mathcal{I}}_0 \to \mathrm{Part}$ supported on the set of χ of this form. If P is a partition of n then define σ_P^1 to be the representation $\sigma_{\overline{\mathcal{P}}}$ of $GL_n(k_F)$ where $\overline{\mathcal{P}} : \overline{\mathcal{I}}_0 \to \mathrm{Part}$ takes the trivial representation to P and everything else to zero. In other words, σ_P^1 is the unipotent representation associated to the partition P. If $\mathbb{1}$ is the trivial representation of $GL_1(k_F)$ then, under the isomorphism of PSH-algebras $R(\mathbb{1}) \xrightarrow{\sim} R^S$ of Proposition 6.9, σ_P^1 corresponds to σ_P°.

Lemma 7.4.   (1) *Every irreducible representation of $GL_n(k_F)$ having a B_1-fixed vector is of the form $\sigma_{\overline{\mathcal{P}}}$ for some $\overline{\mathcal{P}} \in \overline{\mathcal{I}}_1$.*

     (2) *If P is a partition of n, then $\mathrm{red}(\sigma_P^1)$ is irreducible.*

     (3) *If $\overline{\mathcal{P}} \in \overline{\mathcal{I}}_1$ sends each χ to a partition P_χ of degree n_χ, then let P be the partition of n whose parts are the n_χ, let $\overline{M}$ be the corresponding standard*

Levi subgroup of $\overline{G} = GL_n(k_F)$, and let $\overline{Q}$ be a parabolic subgroup with Levi $\overline{M}$. Then

$$\mathrm{red}(\sigma_{\overline{\mathcal{P}}}) = \mathrm{red}(\mathrm{Ind}_{\overline{Q}}^{\overline{G}}(\bigotimes_{\chi} \sigma_{P_\chi}^1)).$$

Proof. If σ is an irreducible representation of $GL_n(k_F)$ having a B_1-fixed vector, then it has non-trivial U-invariants, on some subrepresentation of which $T = B/U$ acts as $\chi = \bigotimes_{i=1}^n \chi_i$ with χ_i having values in μ_{l^a}. So σ is a subquotient of $\mathrm{Ind}_B^G \chi$ and is therefore of the required form, proving part 1.

Part 2 follows from the discussion in section 3 of [Jam90].

Part 3 is immediate from the definition of $\sigma_{\overline{\mathcal{P}}}$ and the observation that if χ takes values in μ_{l^a} then its mod λ reduction is trivial. $\qquad\square$

The representation $\mathrm{Ind}_{\overline{Q}}^{\overline{G}}(\bigotimes \sigma_{P_\chi}^1)$ appearing in part 3 of the lemma decomposes as a direct sum of irreducible representations of the form $\sigma_{P'}$ for partitions P' of n. More specifically, from the isomorphism of PSH-algebras $R(\mathbb{1}) \xrightarrow{\sim} R^S$ of Proposition 6.9 we obtain:

$$(11) \qquad \dim \mathrm{Hom}_{\overline{G}}(\sigma_{P'}, \mathrm{Ind}_{\overline{Q}}^{\overline{G}}(\bigotimes_{\chi} \sigma_{P_\chi}^1)) = \dim \mathrm{Hom}_{S_n}(\sigma_{P'}^\circ, \mathrm{Ind}_{S_P}^{S_n}(\bigotimes_{\chi} \sigma_{P_\chi}^\circ)).$$

Here S_P, $\sigma_{P'}^\circ$ and $\sigma_{P_\chi}^\circ$ are as in section 6.1.

We will need to compute the Mackey decomposition

$$\mathrm{Res}_{S_P}^{S_n} \mathrm{Ind}_{S_Q}^{S_n} \mathrm{sgn}$$

for pairs of partitions P and Q of degree n, and for this we introduce some notation:

Definition 7.5. Let $P, Q \in \mathrm{Part}$ of degree n. A (P,Q)-**bipartition** is a matrix $A = (a(i,j))_{i,j}$ of non-negative integers (with $i, j \in \mathbb{N}$) such that:

- all but finitely many $a(i,j)$ are zero;
- for each i, the sum $\sum_j a(i,j)$ of the entries of the ith row is $P(i)$;
- for each j, the sum $\sum_i a(i,j)$ of the entries of the jth column is $Q(j)$.

The ith row of A determines a partition P_i of $P(i)$. We define the **weight** of A to be the sequence of partitions $(P_1, P_2, \ldots)$.

If $(P_i)_i$ is a finite sequence of partitions and P is the partition formed by their degrees, then define $\mathrm{Bip}\,((P_i)_i, Q)$ to be the number of (P,Q)-bipartitions of weight $(P_i)_i$.

If P is a partition of n, then let $T_{P,i}$ be the set

$$\left\{ 1 + \sum_{j=1}^{i-1} P(j), 2 + \sum_{j=1}^{i-1} P(j), \ldots, \sum_{j=1}^{i} P(j) \right\}$$

(with the convention that this is empty if the first term is greater than the last), so that $\{1, \ldots, n\}$ is the disjoint union of the $T_{P,i}$; write T_P for the sequence $(T_{P,i})_i$. In the left action of S_n on the set of partitions of $\{1, \ldots, n\}$ into disjoint subsets, S_P is the stabiliser of T_P.

Lemma 7.6. *Let P and Q be partitions of n. There is a bijection between the double coset set $S_P \backslash S_n / S_Q$ and the set of all (P,Q)-bipartitions.*

Proof. This is standard; let us just recall the construction. If $g \in S_n$, define a matrix $A_g = (A_g(i,j))$ by

$$A_g(i,j) = \#(T_{P,i} \cap gT_{Q,j}).$$

Then A_g is a (P,Q)-bipartition that only depends on the double coset $S_P g S_Q$, and the map $S_P g S_Q \mapsto A_g$ gives the required bijection. $\square$

Proposition 7.7. *Let P and Q be partitions of n. Then we have:*

$$\mathrm{Res}^{S_n}_{S_P}(\pi^\circ_Q) \cong \bigoplus_{(P_i)_i} \left(\mathrm{Bip}\,((P_i)_i, Q) \cdot \bigotimes_i \pi^\circ_{P_i} \right)$$

where the sum runs over all sequences of partitions $(P_1, P_2, \ldots)$ with $\deg P_i = P(i)$ and, for an integer a and representation ρ, $a \cdot \rho$ denotes the direct sum of a copies of ρ.

Proof. By definition, $\pi^\circ_Q = \mathrm{Ind}^{S_n}_{S_Q}(\mathrm{sgn})$ and $\pi^\circ_{P_i} = \mathrm{Ind}^{S_{P(i)}}_{S_{P_i}}(\mathrm{sgn})$ for each i. The formula follows from Mackey's theorem upon observing that, if $S_P g S_Q$ is the double coset corresponding to a (P,Q)-bipartition of weight $(P_i)_i$, then $S_P \cap S^g_Q$ is conjugate (in S_P) to the subgroup $\prod_i S_{P_i} \subset \prod_i S_{P(i)} = S_P$. $\square$

7.3. **Deformation rings at distinguished points.** Let $(\overline{M}, \overline{\rho})$ be a representation of G_F over $\mathbb{F}$ such that $\tilde{P}_F$ acts trivially, and that $(\overline{e}_i)_i$ is a basis for $\overline{M}$.

Definition 7.8. Say that $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$ is **distinguished** if:
- it is standard, with generalized eigenspace decomposition $\overline{M} = \overline{M}_1 \oplus \ldots \oplus \overline{M}_r$ for $\overline{\rho}(\phi)$ (we thus adopt the notation of Definition 7.2);
- for each i, $\overline{M}_i$ is stable under $\overline{\rho}(\sigma)$;
- for each i, the minimal polynomial of $\overline{\rho}(\sigma)$ acting on $\overline{M}_i$ is $(X-1)^{n_i}$.

Lemma 7.9. *Suppose that $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$ is distinguished and that $(M, \rho, (e_i)_i)$ is a standard lift to some $A \in \mathcal{C}_\mathcal{O}$. Let $M = M_1 \oplus \ldots \oplus M_r$ be the decomposition of Definition 7.2. Then $\rho(\sigma)$ preserves each M_i.*

Proof. Let $\Sigma = \rho(\sigma) \in \mathrm{End}(M)$ and let $\Phi = \rho(\phi) \in \mathrm{End}(M)$. Let Φ_i be the image of Φ in $\mathrm{End}(M_i)$; then by assumption $\Phi = \bigoplus_{i=1}^r \Phi_i$. Let Σ_{ij} be the image of Σ in $\mathrm{Hom}(M_i, M_j)$; we must show that $\Sigma_{ij} = 0$ for $i \neq j$.

Let I be the ideal of A generated by the matrix entries of Σ_{ij} (with respect to the basis (e_i)) for $i \neq j$. We will show that $I = \mathfrak{m}_A I$ and hence, by Nakayama, that $I = 0$, as required. Write

$$\Sigma^q = (1 + (\Sigma - 1))^q$$

$$= 1 + q(\Sigma - 1) + \sum_{s \geq 2} \binom{q}{s} (\Sigma - 1)^s.$$

As $q - 1 \in \mathfrak{m}_A$, $\binom{q}{s} \in \mathfrak{m}_A$ for $2 \leq s \leq n$ (using that l is quasi-banal), and $(\Sigma - 1)^n \equiv 0 \bmod \mathfrak{m}_A$, we see that

$$(\Sigma^q)_{ij} \equiv \Sigma_{ij} \quad \bmod \mathfrak{m}_A I$$

for $i \neq j$. From the equation $\Phi\Sigma = \Sigma^q\Phi$ we deduce:

$$\Phi_i \Sigma_{ij} = (\Sigma^q)_{ij} \Phi_j$$

$$\equiv \Sigma_{ij} \Phi_j \quad \bmod \mathfrak{m}_A I.$$

If P_i is the characteristic polynomial of Φ_i, then $P_i(\Phi_j)$ is invertible for $i \neq j$ (as the reductions mod $\mathfrak{m}_A$ of P_i and P_j are coprime). But we have

$$0 = P_i(\Phi_i)\Sigma_{ij} \equiv \Sigma_{ij} P_i(\Phi_j) \mod \mathfrak{m}_A I$$

and so $\Sigma_{ij} \equiv 0$ mod $\mathfrak{m}_A I$ as claimed. $\qquad\square$

If χ is a representation of $k_F^\times$ with image in μ_{l^a}, then we regard χ as an element of $\mathcal{I}_0$ via the canonical surjection $I_F \twoheadrightarrow k_F^\times$. Let $\mathcal{I}_1 \subset \mathcal{I}$ be the set of $\mathcal{P} : \mathcal{I} \to \text{Part}$ supported on such χ; note that $\mathcal{I}_1$ can be identified with the set $\overline{\mathcal{I}}_1$ from the last section. For convenience, we pick an enumeration $\chi_1, \ldots, \chi_{l^a}$ of the characters $k_F^\times \to \mu_{l^a}$; thus an element of $\mathcal{I}_1$ can be regarded as a sequence $(P_1, \ldots, P_{l^a})$ of partitions.

To compute $R^\square(\overline{\rho}, \tau_\mathcal{P})$ for a distinguished $\overline{\rho}$ and for $\mathcal{P} \in \mathcal{I}_1$, first note that Lemma 7.3 allows us to reduce to the case in which there is a single $\overline{M}_i$. We then have:

Proposition 7.10. *Suppose that $\overline{\rho}(\sigma)$ has minimal polynomial $(X-1)^n$. Let $\mathcal{P} \in \mathcal{I}$. If $\mathcal{P} \notin \mathcal{I}_1$ then $R^\square(\overline{\rho}, \tau_\mathcal{P}) = 0$. If $\mathcal{P} \in \mathcal{I}_1$ corresponds to a sequence $(P_1, \ldots, P_{l^a})$ of partitions, then:*

- *$R^\square(\overline{\rho}, \tau_\mathcal{P}) = 0$ if any P_i has more than one part (i.e. if $P_i(2) > 0$ for some i);*
- *$R^\square(\overline{\rho}, \tau_\mathcal{P})$ is formally smooth of relative dimension n^2 over $\mathcal{O}$ if each P_i has only one part.*

The special fibre $R^\square(\overline{\rho}) \otimes \mathbb{F}$ has a single minimal prime.

Proof. If $\rho : G_F \to GL_n(\mathcal{O}')$ has reduction isomorphic to $\overline{\rho}$, with $\mathcal{O}'$ the ring of integers in a finite extension E'/E, then the minimal polynomial $f(X)$ of $\rho(\sigma)$ is congruent to $(X - 1)^n$ modulo the maximal ideal of $\mathcal{O}'$. Moreover, its roots are $(q^d - 1)$th roots of unity for some $d \leq n$. As l is quasi-banal, it follows that the roots are l^ath roots of unity. We deduce that $R^\square(\overline{\rho}, \tau_\mathcal{P}) = 0$ if $\mathcal{P} \notin \mathcal{I}_1$.

Suppose that $\mathcal{P} \in \mathcal{I}_1$ corresponds to $(P_1, \ldots, P_{l^a})$. If some P_i has more than one part, then the minimal polynomial of $\rho(\sigma)$ in a lift of $\overline{\rho}$ of type $\tau_\mathcal{P}$ would have degree $< n$. Therefore there are no such lifts and $R^\square(\overline{\rho}, \tau_\mathcal{P}) = 0$ in this case.

Suppose now that each P_i has exactly one part, so that $P_i(1) = n_i$. Let R be the quotient of $R^\square(\overline{\rho})$ obtained by demanding that the characteristic polynomial of $\rho(\sigma)$ is

$$f_\mathcal{P}(X) = \prod_i (X - \chi_i(\sigma))^{n_i}.$$

Then $R^\square(\overline{\rho}, \tau_\mathcal{P})$ is the maximal reduced, l-torsion free quotient of R and in fact we will show that R is formally smooth over $\mathcal{O}$, so that $R = R^\square(\overline{\rho}, \tau_\mathcal{P})$.

Let $(\zeta_1, \ldots, \zeta_n)$ be a tuple of elements of $\mathcal{O}^\times$ in which each $\chi_i(\sigma)$ appears precisely n_i times, so that $f_\mathcal{P}(X) = \prod_{i=1}^n (X - \zeta_i)$.

Choose a basis $\overline{e}_1, \ldots, \overline{e}_n$ for $\overline{M}$ with respect to which the action of G_F is given by $\overline{\rho}$. Conjugating $\overline{\rho}$ if necessary, we may assume that

$$\overline{e}_i = (\overline{\rho}(\sigma) - 1)^{i-1}\overline{e}_1$$

for $1 \leq i \leq n$.

Let $(M, \rho, (e_i)_i)$ be a lift of $(\overline{M}, \overline{\rho}, (\overline{e}_i)_i)$ to some $A \in \mathcal{C}_\mathcal{O}$ and suppose that the characteristic polynomial of $\rho(\sigma)$ is $f_\mathcal{P}(X)$ (regarded as an element of $A[X]$ via the

structure map $\mathcal{O} \to A$). Write $\Sigma = \rho(\sigma) \in \mathrm{End}(M)$. Define $f_1, \ldots, f_n \in M$ by

$$f_1 = e_1$$
$$f_2 = (\Sigma - \zeta_1)e_1$$
$$f_3 = (\Sigma - \zeta_1)(\Sigma - \zeta_2)e_1$$
$$\vdots$$
$$f_n = (\Sigma - \zeta_1)(\Sigma - \zeta_2)\ldots(\Sigma - \zeta_{n-1})e_1.$$

Then $f_1, \ldots, f_n$ is a basis of M in which the matrix of Σ is:

$$\begin{pmatrix} \zeta_1 & 0 & 0 & 0 & \ldots \\ 1 & \zeta_2 & 0 & 0 & \ldots \\ 0 & 1 & \zeta_3 & 0 & \ldots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & \zeta_n \end{pmatrix}.$$

Let S be the maximal quotient of R on which Σ has this form. Since the formation of the f_i from the e_i is functorial, we have a morphism $S \to R$ in $\mathcal{C}_{\mathcal{O}}^{\wedge}$ that is easily seen to be formally smooth. To see that S is formally smooth over $\mathcal{O}$, I claim that for every $m \in M$ there is a unique $\Phi \in \mathrm{End}(M)$ such that $\Phi(f_1) = m$ and $\Phi\Sigma = \Sigma^q\Phi$. Indeed, for each i we must have

$$\Phi(f_i) = \Phi(\Sigma - \zeta_1)\ldots(\Sigma - \zeta_{i-1})f_1$$
$$= (\Sigma^q - \zeta_1)\ldots(\Sigma^q - \zeta_{i-1})\Phi(f_1)$$
$$= (\Sigma^q - \zeta_1)\ldots(\Sigma^q - \zeta_{i-1})m,$$

and the endomorphism Φ defined by this formula works. Therefore lifting $\overline{\rho}(\phi)$ to an automorphism $\rho(\phi)$ of M such that $\rho(\phi)\Sigma\rho(\phi)^{-1} = \Sigma^q$ is the same as giving a single element of M lifting $\overline{\rho}(f_1)$, and we see that S is formally smooth of dimension n over $\mathcal{O}$. Thus R, and hence also $R^{\square}(\overline{\rho}, \tau_{\mathcal{P}})$, is formally smooth over $\mathcal{O}$ as required.

For the statement about the special fibre, simply note that $R \otimes \mathbb{F}$, as a quotient of $R^{\square}(\overline{\rho}) \otimes \mathbb{F}$, is independent of the choice of $\mathcal{P}$. $\square$

Corollary 7.11. *Suppose that $\overline{\rho}$ is distinguished and that the generalised eigenspaces of $\overline{\rho}(\phi)$ have dimensions $n_1 \geq n_2 \geq \ldots \geq n_r$. Let Q be the partition of n with $Q(i) = n_i$ for $1 \leq i \leq r$. Let $\mathcal{P} \in \mathcal{I}$. If $\mathcal{P} \notin \mathcal{I}_1$ then $R^{\square}(\overline{\rho}, \tau_{\mathcal{P}}) = 0$. Otherwise, suppose that $\mathcal{P}$ corresponds to the sequence of partitions $(P_i)_i$, and suppose (without loss of generality) that $\deg P_1 \geq \deg P_2 \geq \ldots$. Let P be the partition of n with $P(i) = \deg P_i$. Then $R^{\square}(\overline{\rho}) \otimes \mathbb{F}$ has a unique minimal prime $\mathfrak{p}$ and*

$$Z(R^{\square}(\overline{\rho}, \tau_{\mathcal{P}}) \otimes \mathbb{F}) = \mathrm{Bip}\left((P_i)_i, Q\right) \cdot [\mathfrak{p}].$$

Proof. We combine Lemmas 7.3, 7.9 and Proposition 7.10. Let $\overline{M}_j$ $(1 \leq j \leq r)$ be the generalised eigenspaces of $\overline{\rho}(\phi)$ on $\overline{M}$ and let $\overline{\rho}_j$ be the representation of G_F on $\overline{M}_j$ for each j. Then we have that $R^{\square}(\overline{\rho})$ is formally smooth over $\widehat{\bigotimes}_j R^{\square}(\overline{\rho}_j)$, by Lemma 7.3. That $R^{\square}(\tau_{\mathcal{P}})$ is zero if $\mathcal{P} \notin \mathcal{I}_1$ is now clear.

If $\mathcal{P} \in \mathcal{I}_1$ corresponds to the sequence $(P_i)_i$, then the irreducible components of $\mathrm{Spec}\, R^{\square}(\overline{\rho}, \tau_{\mathcal{P}})$ are all formally smooth with the same special fibre. The number of such irreducible components is the number of sequences $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r)$ where:

- for $j = 1, \ldots, r$, $\mathcal{P}_j \in \mathcal{I}_1$ has degree n_j;

- each $\mathcal{P}_j(\chi_i)$ consists of a single part (that is, $\mathcal{P}_j(\chi_i)(1) = d_{ij}$ for some non-negative integer d_{ij}, and $\mathcal{P}_j(\chi_i)(2) = 0$);
- for each i, the sequence $(d_{i1}, d_{i2}, \ldots, d_{ir})$ is a reordering of $P_i(1), \ldots, P_i(r)$.

Indeed, such a sequence gives rise to the irreducible component

$$\mathrm{Spec} \,\widehat{\bigotimes_j} R^\square(\overline{\rho}_j, \tau_{\mathcal{P}_j})$$

of

$$\mathrm{Spec} \,\widehat{\bigotimes_j} R^\square(\overline{\rho}_j),$$

and hence of $\mathrm{Spec}\, R^\square(\overline{\rho})$, that has type $\tau_{\mathcal{P}}$, and all irreducible components have this form.

But now $(d_{ij})_{i,j}$ is a (P, Q)-bipartition of type $(P_i)_i$ and we see that the number of irreducible components of $R^\square(\overline{\rho}, \tau_{\mathcal{P}})$ is the number of (P, Q)-bipartitions of type $(P_i)_i$. Since all the irreducible components are formally smooth with the same special fibre, we get the claimed formula. $\qquad\square$

Let $\overline{\mathfrak{X}}_1$ be the closed subscheme of $\overline{\mathfrak{X}}$ on which Σ is unipotent. Let $\mathfrak{X}_1$ be the connected component of $\mathfrak{X}$ containing $\overline{\mathfrak{X}}$.

Lemma 7.12. *Every irreducible component of $\overline{\mathfrak{X}}_1$ contains a point $x \in \overline{\mathfrak{X}}(\mathbb{F})$ such that $\overline{\rho}_x$ is distinguished (possibly after enlarging $\mathbb{F}$). If $x \in \overline{\mathfrak{X}}(\mathbb{F})$ is such that $\overline{\rho}_x$ is distinguished, then x lies on a unique irreducible component of $\overline{\mathfrak{X}}$.*

Proof. The irreducible components of $\overline{\mathfrak{X}}$ are precisely the closures of the preimages under π_Σ of conjugacy classes of Σ in $GL_n(\overline{\mathbb{F}})$.[7] If Σ is unipotent, then (using that l is quasi-banal and so $\binom{q}{i} = 0 \mod l$ for $2 \leq i \leq n$):

$$\Sigma^q = (1 + (\Sigma - 1))^q$$

$$= 1 + q(\Sigma - 1) + \sum_{i=2}^{n} \binom{q}{i}(\Sigma - 1)^i$$

$$= 1 + (\Sigma - 1) = \Sigma.$$

Thus (for unipotent Σ) the equation $\Phi\Sigma\Phi^{-1} = \Sigma^q$ is equivalent to Φ commuting with Σ. But then for each unipotent $\Sigma \in GL_n(\overline{\mathbb{F}})$ it is straightforward (using Jordan normal form) to choose a $\Phi \in GL_n(\overline{\mathbb{F}})$ commuting with Σ such that the representation $\overline{\rho}_x$ attached to the point $x = (\Phi, \Sigma)$ of $\overline{\mathfrak{X}}$ is distinguished; possibly enlarging $\mathbb{F}$, we can assume that $\Phi \in GL_n(\mathbb{F})$.

The second assertion follows from the last part of Proposition 7.10. $\qquad\square$

7.4. **Comparison of multiplicities.** Continue to assume that l is quasi-banal. Recall that (given a choice of generator σ of tame inertia) we have defined $\mathrm{cyc}_{\mathrm{ft}} : R^1_E(GL_n(k_F)) \to \mathcal{Z}(\mathfrak{X})$.

---

[7]See the proof of Theorem 2.5.

Theorem 7.13. *There is a unique map* $\overline{\mathrm{cyc}}_{\mathrm{ft}} : R^1_{\mathbb{F}}(GL_n(k_F)) \to \mathcal{Z}(\overline{\mathfrak{X}})$ *such that the diagram*

(12)
$$
\begin{array}{ccc}
R^1_E(GL_n(k_F)) & \xrightarrow{\;\mathrm{cyc}_{\mathrm{ft}}\;} & \mathcal{Z}(\mathfrak{X}) \\
{\scriptstyle\mathrm{red}}\downarrow & & {\scriptstyle\mathrm{red}}\downarrow \\
R^1_{\mathbb{F}}(GL_n(k_F)) & \xrightarrow{\;\overline{\mathrm{cyc}}_{\mathrm{ft}}\;} & \mathcal{Z}(\overline{\mathfrak{X}})
\end{array}
$$

commutes.

Proof. As explained in section 7.1, this implies a similar statement with $\mathfrak{X}$ replaced by $\mathrm{Spec}\, R^{\square}(\overline{\rho})$ for any continuous $\overline{\rho} : G_F \to GL_n(k_F)$ such that $\overline{\rho}|_{\tilde{P}_F}$ is trivial. Moreover, by Proposition 7.1 and Lemma 7.12, it suffices to prove that, for $\overline{\rho}$ distinguished, there is a map $\overline{\mathrm{cyc}} : R^1_{\mathbb{F}}(GL_n(k_F)) \to \mathcal{Z}(\overline{R}^{\square}(\overline{\rho}))$ such that

(13)
$$
\begin{array}{ccc}
R^1_E(GL_n(k_F)) & \xrightarrow{\;\mathrm{cyc}\;} & \mathcal{Z}(R^{\square}(\overline{\rho})) \\
{\scriptstyle\mathrm{red}}\downarrow & & {\scriptstyle\mathrm{red}}\downarrow \\
R^1_{\mathbb{F}}(GL_n(k_F)) & \xrightarrow{\;\overline{\mathrm{cyc}}\;} & \mathcal{Z}(\overline{R}^{\square}(\overline{\rho}))
\end{array}
$$

commutes. (Although we work with the whole $R_E(GL_n(\mathcal{O}_F))$ in section 7.1, the arguments apply just as well with $R^1_E(GL_n(k_F))$, using that $\mathrm{red} : R^1_E(GL_n(k_F)) \to R^1_{\mathbb{F}}(GL_n(k_F))$ is surjective in the quasi-banal case.)

So suppose that $\overline{\rho}$ is distinguished, and that the generalized eigenspaces of $\overline{\rho}(\phi)$ have dimensions $n_1 \geq n_2 \geq \ldots \geq n_r$, giving a partition $Q = (n_1, n_2, \ldots, n_r)$ of n. First we make the definition of the cycle map explicit. If $\overline{\mathcal{P}} \in \overline{\mathcal{I}}_1$, then we have (identifying an element $\mathcal{P}' \in \mathcal{I}_1$ with an element $\overline{\mathcal{P}}'$ of $\overline{\mathcal{I}}_1$):

$$
\mathrm{cyc} : \sigma_{\overline{\mathcal{P}}} \mapsto \sum_{\mathcal{P}' \in \mathcal{I}_1} \dim \mathrm{Hom}(\sigma_{\overline{\mathcal{P}}}, \pi_{\overline{\mathcal{P}}'}) Z(R^{\square}(\overline{\rho}, \tau_{\mathcal{P}'}^{\vee})).
$$

Note that $R^{\square}(\overline{\rho}, \tau_{\mathcal{P}'}) = 0$ for $\mathcal{P}' \notin \mathcal{I}_1$, and that we have (for convenience) rearranged the position of the dual occurring in Definition 4.4. If $\mathfrak{p}$ is the unique minimal prime of $R^{\square}(\overline{\rho}) \otimes \mathbb{F}$, then we find (by Corollary 7.11) that

$$
\mathrm{red} \circ \mathrm{cyc} : \sigma_{\overline{\mathcal{P}}} \mapsto [\mathfrak{p}] \cdot \sum_{\mathcal{P}'} \dim \mathrm{Hom}(\sigma_{\overline{\mathcal{P}}}, \pi_{\overline{\mathcal{P}}'}) \mathrm{Bip}\left(\left(\mathcal{P}'(\chi_i)\right)_i, Q\right).
$$

Now, if $\overline{\mathcal{P}}$ takes χ_1 (the trivial representation) to the partition P of n, then we see that

$$
\mathrm{red} \circ \mathrm{cyc} : \sigma_{\overline{\mathcal{P}}} \mapsto [\mathfrak{p}] \cdot \dim \mathrm{Hom}(\sigma_P^{\circ}, \pi_Q^{\circ})
$$

and so we must have

$$
\overline{\mathrm{cyc}}(\mathrm{red}(\sigma_{\overline{\mathcal{P}}})) = [\mathfrak{p}] \cdot \dim \mathrm{Hom}(\sigma_P^{\circ}, \pi_Q^{\circ}).
$$

By Lemma 7.4, the $\mathrm{red}(\sigma_{\overline{\mathcal{P}}})$ for $\mathcal{P}$ supported on the trivial representation are all irreducible, and are a basis for $R^1_{\mathbb{F}}(GL_n(k_F))$; there is therefore a unique map $\overline{\mathrm{cyc}}$ defined by the above equation for such $\mathcal{P}$, and we must show that it makes diagram (13) commute. Using Lemma 7.4 and the subsequent equation (11), we see that it suffices to show the following. If $\mathcal{P} \in \overline{\mathcal{I}}_1$ has $\mathcal{P}(\chi_i) = P_i$, and P is the partition corresponding to $(\deg P_1, \deg P_2, \ldots)$, then

$$
\sum_{P'} \dim \mathrm{Hom}_{S_n}(\sigma_{P'}^{\circ}, \pi_Q^{\circ}) \dim \mathrm{Hom}_{S_n}\left(\sigma_{P'}^{\circ}, \mathrm{Ind}_{S_P}^{S_n}\left(\bigotimes \sigma_{P_i}^{\circ}\right)\right)
$$

is equal to

$$\sum_{\mathcal{P}'} \dim \operatorname{Hom}(\sigma_{\overline{\mathcal{P}}}, \pi_{\overline{\mathcal{P}}'}) \operatorname{Bip}\left((\mathcal{P}'(\chi_i))_i, Q\right),$$

where the first sum is over partitions P' of n and the second is over $\mathcal{P}' \in \mathcal{I}_1$. Indeed, the first displayed equation is the value of $\overline{\operatorname{cyc}}(\operatorname{red}(\sigma_{\overline{\mathcal{P}}}))$, and the second is the value of $\operatorname{red}(\operatorname{cyc}(\sigma_{\overline{\mathcal{P}}}))$.

But

$$\sum_{P'} \dim \operatorname{Hom}(\sigma_{P'}^{\circ}, \pi_Q^{\circ}) \dim \operatorname{Hom}\left(\sigma_{P'}^{\circ}, \operatorname{Ind}_{S_P}^{S_n}\left(\bigotimes \sigma_{P_i}^{\circ}\right)\right)$$

$$= \dim \operatorname{Hom}\left(\pi_Q^{\circ}, \operatorname{Ind}_{S_P}^{S_n}\left(\bigotimes \sigma_{P_i}^{\circ}\right)\right)$$

$$= \sum_{(P_i')_i} \operatorname{Bip}\left((P_i')_i, Q\right) \dim \operatorname{Hom}_{S_P}\left(\bigotimes \pi_{P_i'}^{\circ}, \bigotimes \sigma_{P_i}^{\circ}\right)$$

$$= \sum_{\mathcal{P}'} \operatorname{Bip}\left((\mathcal{P}'(\chi_i))_i, Q\right) \dim \operatorname{Hom}(\sigma_{\overline{\mathcal{P}}}, \pi_{\overline{\mathcal{P}}'})$$

as required. The sum on the third line is over sequences of partitions $(P_i')_i$ with $\deg P_i' = \deg P_i$, and to go from the second to the third line we have used Proposition 7.7. The sum on the fourth line is over $\mathcal{P}' \in \mathcal{I}_1$ and to go from the third to the fourth line we have used Corollary 6.10. $\qquad\square$

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