**Math 162 – final exam**

Instructor: Jack Shotton.

March 13th/17th 2017.

Time available: 120 minutes.

This exam is marked out of 120, and counts for 50% of the course grade.

Write neatly. Start with the questions you know how to do, and aim to spend about 20 minutes on each question.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. Let $f$ be a bounded function on $[a, b]$.

   (a) (12 points) Prove that, if $f$ is continuous, then $f$ is integrable.

   *You may assume that a continuous function on $[a, b]$ is uniformly continuous, and any other results from class that you need.*

   (b) (8 points) Suppose that $f$ is integrable, $\int_a^b f = 0$, and that $g$ is a function such that

   $$0 \leq g(x) \leq f(x)$$

   for all $x \in [a, b]$.

   Prove that $g$ is integrable and that $\int_a^b g = 0$.

2. If $f$ is a continuous function on $[a, b]$, let $F$ be the function defined by

   $$F(x) = \int_a^x f$$

   for $x \in [a, b]$.

   (a) (4 points) State the first fundamental theorem of calculus.

   (b) (6 points) Give an example of an integrable function $f$ on $[-1, 1]$ such that $F$ is not differentiable at 0. *Briefly justify that your example works.*

   (c) (6 points) Give an example of an integrable function $f$ on $[-1, 1]$ such that $f$ is not continuous at 0, but $F$ is differentiable at 0. *Briefly justify that your example works.*

   (d) (4 points) Give an example of an integrable function $f$ on $[-1, 1]$ such that $f$ is discontinuous at infinitely many points, but $F$ is differentiable at every $x \in (-1, 1)$. *You do* not *have to justify that your example works.*

3. (a) For $n \geq 1$, let

$$I_n = \int_0^\infty \frac{1}{(1+x^2)^n}\,dx.$$

You can assume that this improper integral converges.

i. (2 points) Find $I_1$.

ii. (8 points) By writing

$$\frac{1}{(1+x^2)^n} = \frac{1+x^2}{(1+x^2)^{n+1}}$$

and using integration by parts, prove that

$$I_{n+1} = \frac{2n-1}{2n} I_n$$

for $n \geq 1$.

iii. (2 points) Find $I_4$.

(b) (8 points) Find the arc length of the parametrised curve $(t - \sin(t), 1 - \cos(t))$ for $t$ between 0 and $2\pi$.

4. (a) (2 points) Define what it means for a sequence $(a_n)_n$ to converge.

(b) (8 points) Prove that a bounded, decreasing sequence converges.

(c) Let $a_1 = 2$ and $a_{n+1} = \frac{2a_n+1}{3}$ for all $n \geq 1$.

i. (3 points) Prove that $a_n \geq 1$ for all $n \geq 1$.

ii. (4 points) Prove that $(a_n)_n$ converges.

iii. (3 points) Find

$$\lim_{n\to\infty} a_n.$$

5. (a) (8 points) Prove that, if a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

   (b) For each of the following series, say whether or not it converges, and give a brief proof:

      i. (4 points)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}};$$

      ii. (4 points)
$$\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}};$$

      iii. (4 points)
$$\sum_{n=2}^{\infty} \frac{1}{\log(n)^{\log(n)}}.$$

6. (a) (4 points) Find the $n$th Taylor polynomial of $e^x$ at 0.

   (b) (8 points) Using Taylor's theorem, prove that
$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

   (c) (8 points) By differentiating the equation
$$\frac{1}{1-x} = 1 + x + \ldots + x^{n-1} + \frac{x^n}{1-x},$$
   prove that if $|x| < 1$ then
$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} a_n x^n$$
   for certain numbers $a_n$ which you should find.