**Math 163 section 21 – final**

Instructor: Jack Shotton.

Wednesday 7th June 2017, 10:30–12:30.

Time available: 2 hours.

This exam is marked out of 120, and counts for 50% of the course grade.

Write neatly. Start with the questions you know how to do.

Notation: $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{N}$ the natural numbers $\{1, 2, 3, \ldots\}$.

1. (a) (5 points) Let $f(x) = 0$ for all $x$. Find a sequence of functions $f_n$ on $[0, 1]$ such that $f_n$ converges to $f$ pointwise but not uniformly on $[0, 1]$.

   Prove that your example does not converge uniformly.

   (b) (8 points) Suppose that $f_n$ and $f$ are functions on $[a, b]$ such that $f_n \to f$ uniformly on $[0, 1]$ and such that every $f_n$ is integrable.

   Prove that $f$ is integrable.

   (c) (7 points) For $x \in \mathbb{R}$, let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}.$$

   Prove that $f$ is integrable on $[a, b]$ for any $b > a \geq 0$ and find

$$\int_0^1 f(x)dx.$$

2. Express the following complex numbers as simply as possible, showing your working.

   (a) (4 points) $|\exp((2+i)^2)|$;

   (b) (4 points) $\cos(i \log(2))$.

   (c) (4 points) $\alpha + \beta + \gamma$ where $\alpha$, $\beta$ and $\gamma$ are the roots of

$$z^3 - 5z^2 + 9z - 3.$$

   *Do* not *try to calculate $\alpha$, $\beta$ and $\gamma$ individually.*

   (d) (4 points) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ where $\alpha$, $\beta$ and $\gamma$ are as in part (c).

   (e) (4 points) $1 - w + w^2 - w^3 + w^4$ where $w = \exp(i\pi/5)$. *Hint: consider $w^5 + 1$.*

3. (a) (10 points) Prove from the definition that $[0, 1]$ is compact.
   (b) Let $X \subset \mathbb{R}^2$ be a non-empty closed, bounded subset such that, for every point $(x, y) \in X$, $y > 0$.
      i. (5 points) Prove that there is a point $(x, y) \in X$ for which $y$ is a minimum.
      ii. (5 points) Is this true if $X$ is only assumed to be closed, not bounded? Justify your answer.

4. (a) (6 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) = f(x + t, y - t)$$

   for fixed $(x, y) \in \mathbb{R}^2$. Using the chain rule, or otherwise, find $g'(0)$ in terms of the partial derivatives of $f$ at $(x, y)$.

   (b) (4 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that

$$f(x + t, y - t) = f(x, y)$$

   for all $(x, y) \in \mathbb{R}^2$, $t \in \mathbb{R}$.
   Use part (a) to prove that

$$D_1 f(x, y) - D_2 f(x, y) = 0$$

   for all $(x, y) \in \mathbb{R}^2$.

   (c) (6 points) Let $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be a differentiable function such that

$$f(\lambda x, \lambda y) = f(x, y) \qquad (1)$$

   for all $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, $\lambda \in \mathbb{R} \setminus \{0\}$.
   Prove that

$$x D_1 f(x, y) + y D_2 f(x, y) = 0 \qquad (2)$$

   for all $(x, y) \in \mathbb{R}^2 \setminus \{0\}$.

   (d) (4 points) Write down an example of a non-constant function $f$ satisfying equation (1) and check that equation (2) holds for your chosen function.

5. Let $A = [0,1] \times [0,1] \subset \mathbb{R}^2$.

   (a) (10 points) Prove from the definition of integration that, if $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded function such that $f(x, y) = 0$ for $x \neq y$, then $f$ is integrable on $A$ and

$$\int_A f = 0.$$

   (b) Find counterexamples to the following statements. You do not have to prove that your examples work.

      i. (5 points) If $f$ is an integrable function on $A$ then, for every $x \in [0,1]$, the function $g_x(y) = f(x, y)$ is integrable on $[0,1]$.

      ii. (5 points) If $f$ is an bounded function on $A$ such that, for every $x \in [0,1]$, the function $g_x(y) = f(x, y)$ is integrable on $[0,1]$, then $f$ is integrable on $A$.

6. (a) (6 points) Find

$$\int_A \frac{1}{(x+y)^2}\, dx\, dy$$

   where $A = [1,4] \times [0,2]$, giving your answer as a single logarithm.

   (b) (7 points) Find

$$\int_D y\, dx\, dy$$

   where $D$ is the semicircular region $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, y \geq 0\}$.

   (c) (7 points) Let $f$ be a differentiable function on $\mathbb{R}^2$, with continuous partial derivatives, such that $f(x, 0) = 0$ and $f(x, 1) = x^2$ for all $x \in \mathbb{R}$.
   Let $g(x, y) = D_2 f(x, y)$, and let $A = [a, b] \times [0,1]$. Find

$$\int_A g.$$