EQUIVALENCE CLASSES

Definition. An **equivalence relation** on a set S is a relation $\sim$ on S that is reflexive, symmetric and transitive.

Definition. If S is a set, $\sim$ is an equivalence relation on S, and $a \in S$, then the **equivalence class** $C(a)$ of a is the subset

$$C(a) = \{b \in S : b \sim a\}$$

of S.

Proposition. *Let S be a set and $\sim$ be an equivalence relation on S. Then:*

(1) *if $a \sim b$ then $C(a) = C(b)$;*
(2) *if $C(a) \cap C(b) \neq \emptyset$ then $a \sim b$;*
(3) *the equivalence classes partition S. That is, every element of S is in exactly one equivalence class.*

Proof. (1) Suppose that $a \sim b$. We first show that $C(a) \subset C(b)$. Suppose that $c \in C(a)$. Then $c \sim a$. As $a \sim b$, transitivity gives that $c \sim b$. So $c \in C(b)$, as required.

To show that $C(b) \subset C(a)$, first observe that $b \sim a$ by symmetry. Then the same argument shows that $C(b) \subset C(a)$.

(2) Suppose that $c \in C(a) \cap C(b)$. Then $c \sim a$ and $c \sim b$. By symmetry, $a \sim c$. Combining $a \sim c$ and $c \sim b$ with transitivity gives $a \sim b$, as required.

(3) Every element $x \in S$ is in $C(x)$, because $x \sim x$ by reflexivity. So every element is in at least one equivalence class.

If $x \in S$ is in two equivalence classes $C(a)$ and $C(b)$, then (by part 2) $a \sim b$. But then by part 1, $C(a) = C(b)$. So every $x \in S$ is in at most one distinct equivalence class.

Therefore every $x \in S$ is in exactly one equivalence class, as required.

□