MATH 159 – AXIOMS

A **field** is a set F with operations $+$ and $\times$ such that:

- **AC:** For all $a, b \in F$, $a + b = b + a$;
- **AA:** For all $a, b, c \in F$, $a + (b + c) = (a + b) + c$;
- **AId:** There exists a unique $0 \in F$ such that, for all $a \in F$, $0 + a = a + 0 = a$;
- **AIn:** For all $a \in F$ there exists a unique $-a \in F$ such that $a + (-a) = (-a) + a = 0$.
- **MC:** For all $a, b \in F$, $a \times b = b \times a$;
- **MA:** For all $a, b, c \in F$, $a \times (b \times c) = (a \times b) \times c$;
- **MId:** There exists a unique $1 \in F$ such that, for all $a \in F$, $1 \times a = a \times 1 = a$;
- **MIn:** For all $a \in F$ with $a \neq 0$ there exists a unique $a^{-1} \in F$ such that $a \times a^{-1} = a^{-1} \times a = 0$.
- **D:** For all $a, b, c \in F$, $a \times (b + c) = a \times b + a \times c$ and similarly for $(b + c) \times a$;
- **"1 ≠ 0":** $1 \neq 0$.

Many of the above axioms have redundancies in the light of the commutativity axioms AC and MC, and it's fine to just check one part of them in this case (e.g. you only need to check one of $0 + a = a$ and $a + 0 = a$.) The uniqueness statements are also redundant; for example, if 0 and $0'$ both satisfy the requirement for AId then $0' = 0 + 0' = 0$.

An **ordered field** is a field together with a relation $<$ satisfying the following axioms:

- **O1:** For all $a, b \in F$, precisely one of $a > b$, $a = b$ or $b < a$ is true;
- **O2:** For all $a, b, c \in F$, if $a < b$ and $b < c$ then $a < c$;
- **O3:** For all $a, b, c \in F$, if $a < b$ then $a + c < b + c$;
- **O4:** For all $a, b, c \in F$, if $a < b$ and $0 < c$ then $ac < bc$.