MATH 159 – AXIOMS

A **ring** (e.g. $\mathbb{Z}$) is a set R with operations $+$ and $\times$ such that:

AA: For all $a, b, c \in R$, $a + (b + c) = (a + b) + c$;
MA: For all $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$;
AC: For all $a, b \in R$, $a + b = b + a$;
MC: For all $a, b \in R$, $a \times b = b \times a$;
D: For all $a, b, c \in R$, $a \times (b + c) = a \times b + a \times c$ and similarly for $(b + c) \times a$;
AId: There exists $0 \in R$ such that, for all $a \in R$, $0 + a = a + 0 = a$;
MId: There exists $1 \in R$ such that, for all $a \in R$, $1 \times a = a \times 1 = a$;
AIn: For all $a \in R$ there exists $-a \in R$ such that $a + (-a) = (-a) + a = 0$.

A ring satisfies the **cancellation axiom** if:

C: For all $a, b \in \mathbb{Z}$, if $a \times b = 0$ then $a = 0$ or $b = 0$.

For instance, $\mathbb{Z}$ satisfies the cancellation axiom but $\mathbb{Z}_6$ does not as $\overline{2} \times \overline{3} = \overline{0}$ but $\overline{2}, \overline{3} \neq 0$.

A **field** (e.g. $\mathbb{Q}$) is a ring in which the following two extra axioms hold:

MIn: For all $a \in R$ with $a \neq 0$, there exists $a^{-1} \in R$ with $a \times a^{-1} = a^{-1} \times a = 1$.
"1 ≠ 0": $1 \neq 0$.

Any field satisfies the cancellation axiom.

Many of the above axioms have redundancies in the light of the commutativity axioms AC and MC, and it's fine to just check one part of them in this case (e.g. you only need to check one of $0 + a = a$ and $a + 0 = a$.)

An **ordered ring** (or field) is a ring (or field) together with a relation $<$ satisfying the following axioms:

O1: For all $a, b \in R$, precisely one of $a > b$, $a = b$ or $b < a$ is true;
O2: For all $a, b, c \in R$, if $a < b$ and $b < c$ then $a < c$;
O3: For all $a, b, c \in R$, if $a < b$ then $a + c < b + c$;
O4: For all $a, b, c \in R$, if $a < b$ and $0 < c$ then $ac < bc$.

We have one additional axiom for $\mathbb{Z}$, the **well-ordering principle**:
if $S \subset \mathbb{Z}$ is a non-empty bounded-below subset, then S has a smallest element. Then $\mathbb{Z}$ is the unique ordered ring satisfying the well-ordering principle.

The well-ordering principle is equivalent to the **principle of mathematical induction**: if $S \subset \mathbb{N}$ is a subset such that $1 \in S$ and, for all $n \in S$, $n + 1 \in S$, then $S = \mathbb{N}$.