# MATH 159 – AXIOMS

A **ring** (e.g. $\mathbb{Z}$) is a set $R$ with operations $+$ and $\times$ such that:

**AA:** For all $a, b, c \in R$, $a + (b + c) = (a + b) + c$;
**MA:** For all $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$;
**AC:** For all $a, b \in R$, $a + b = b + a$;
**MC:** For all $a, b \in R$, $a \times b = b \times a$;
**D:** For all $a, b, c \in R$, $a \times (b + c) = a \times b + a \times c$ and similarly for $(b + c) \times a$;
**AId:** There exists $0 \in R$ such that, for all $a \in R$, $0 + a = a + 0 = a$;
**MId:** There exists $1 \in R$ such that, for all $a \in R$, $1 \times a = a \times 1 = a$;
**AIn:** For all $a \in R$ there exists $-a \in R$ such that $a + (-a) = (-a) + a = 0$.

For the integers $\mathbb{Z}$, we also have the cancellation axiom:

**C:** For all $a, b \in \mathbb{Z}$, if $a \times b = 0$ then $a = 0$ or $b = 0$.

A ring in which the cancellation axiom holds is an **integral domain**.
A **field** (e.g. $\mathbb{Q}$) is a ring in which the following two extra axioms hold:

**MIn:** For all $a \in R$ with $a \neq 0$, there exists $a^{-1} \in R$ with $a \times a^{-1} = a^{-1} \times a = 1$.
**"$1 \neq 0$":** $1 \neq 0$.

Many of the above axioms have redundancies in the light of the commutativity axioms AC and MC, and it's fine to just check one part of them in this case (e.g. you only need to check one of $0 + a = a$ and $a + 0 = a$.)
An **ordered ring** (or field) is a ring (or field) together with a relation $<$ satisfying the following axioms:

**O1:** For all $a, b \in R$, precisely one of $a > b$, $a = b$ or $b < a$ is true;
**O2:** For all $a, b, c \in R$, if $a < b$ and $b < c$ then $a < c$;
**O3:** For all $a, b, c \in R$, if $a < b$ then $a + c < b + c$;
**O4:** For all $a, b, c \in R$, if $a < b$ and $0 < c$ then $ac < bc$.

We have one additional axiom for $\mathbb{Z}$, the **well-ordering principle**:
if $S \subset \mathbb{Z}$ is a non-empty bounded-below subset, then $S$ has a smallest element.
We say that that an ordered ring satisfies the **axiom of Archimedes** if the integers are unbounded; that is, there does not exist an element of the ring that is greater than every integer. In this course, we will not encounter any rings in which this is not true; however, they do exist!
The **real numbers**, $\mathbb{R}$, are the unique ordered field such that every non-empty bounded-above subset has a least upper bound. It is true, but we do not prove it in this course, that there exists an ordered field with this property, and that it is unique. If you are curious, see §3.5 in the textbook.