

# HANDOUT: CHAIN RULE PROOF

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $x$ ,  $g: \mathbb{R}^m \rightarrow \mathbb{R}^p$  differentiable at  $y = f(x)$ .

Let  $S = Df(x)$ ,  $T = Dg(y)$ .

WTS:  $D(g \circ f)(x) = T \circ S$ .

Write  $f(x+h) = f(x) + S(h) + E_1(h)|h|$

with  $E_1(0) = 0$ .

By def<sup>n</sup> of derivative,  $\lim_{h \rightarrow 0} E_1(h) = 0$ .  $(\dagger_1)$

Similarly,  $g(y+k) = g(y) + T(k) + E_2(k)|k|$

with  $E_2(k) = 0$ ,  $\lim_{k \rightarrow 0} E_2(k) = 0$ .  $(\dagger_2)$

Let  $k(h) = S(h) + E_1(h)|h|$ .  $(\Delta)$

Lemma (a)  $\lim_{h \rightarrow 0} k(h) = 0$

(b)  $\frac{|k(h)|}{|h|}$  is bounded (for  $|h| < \varepsilon$ ).

Proof (a) Clear, as  $S(h)$  is lts.

(b)  $\frac{|k(h)|}{|h|} \leq \frac{|S(h)|}{|h|} + |E_1(h)|$

$|E_1(h)| \rightarrow 0$  as  $h \rightarrow 0$  so is bounded on  $|h| < \varepsilon$ .

If  $S(h) = a_1 h_1 + \dots + a_n h_n$  then  $\frac{|S(h)|}{|h|} \leq \sqrt{a_1^2 + \dots + a_n^2}$  by Cauchy-Schwarz, so bounded.  $\checkmark$

Now,

$$g(f(x+h)) - g(f(x)) = g(f(x) + k(h)) - g(f(x)) \quad \text{by } *_{1}, \Delta$$

$$= \cancel{g(f(x))} + T(k(h)) + E_2(k(h)) |k(h)| - \cancel{g(f(x))} \quad \text{by } *_{2}$$

$$= T(S(h)) + |h|T(E_1(h)) + |k(h)|E_2(k(h)) \quad \text{by } \Delta$$

+ linearity of  $T$ .

$$\frac{g(f(x+h)) - g(f(x)) - T(S(h))}{|h|} =$$

$$= \underbrace{T(E_1(h))}_{\rightarrow 0 \text{ as } h \rightarrow 0} + \underbrace{E_2(k(h))}_{\rightarrow 0 \text{ as } h \rightarrow 0} \underbrace{\frac{|k(h)|}{|h|}}_{\text{bounded by lemma (b)}}$$

$\rightarrow 0$  as  $h \rightarrow 0$   
by lemma (1) &  
T is dt

$\rightarrow 0$  as  $h \rightarrow 0$   
by (2) and lemma (a)

$$\rightarrow 0 \quad \text{as } h \rightarrow 0.$$

So  $D(g \circ f)(x) = T \circ S$  as required. □

Idea:  $g(f(x+h)) = g(f(x) + S(h) + \text{TINY})$

$$= g(f(x)) + T(S(h)) + \text{TINY} + \text{TINY}$$

$$= g(f(x)) + T(S(h)) + \text{TINY}$$

$$(\text{as } T(\text{TINY}) + \text{TINY} = \text{TINY})$$

& proof is just carefully keeping track of "TINY".