AXIOMS FOR $\mathbb{R}$

The following are the **axioms of arithmetic**

- **AdditionAssociative:** For all $a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$;
- **MultiplicationAssociative:** For all $a, b, c \in \mathbb{R}$, $a \times (b \times c) = (a \times b) \times c$;
- **Add.Commutative:** For all $a, b \in \mathbb{R}$, $a + b = b + a$;
- **Mult.Commutative:** For all $a, b \in \mathbb{R}$, $a \times b = b \times a$;
- **Distributive:** For all $a, b, c \in \mathbb{R}$, $a \times (b + c) = a \times b + a \times c$ and similarly for $(b + c) \times a$;
- **AdditiveIdentity:** There exists $0 \in \mathbb{R}$ such that, for all $a \in \mathbb{R}$, $0 + a = a + 0 = a$;
- **MultiplicativeIdentity:** There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and, for all $a \in \mathbb{R}$, $1 \times a = a \times 1 = a$;
- **Add.Inverse:** For all $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$.
- **Mult.Inverse:** For all $a \in \mathbb{R}$ with $a \neq 0$, there exists $a^{-1} \in \mathbb{R}$ with $a \times a^{-1} = a^{-1} \times a = 1$.

These axioms are not on their own sufficient to describe $\mathbb{R}$ — for example, it is impossible to prove from these axioms that $1 + 1 \neq 0$, or that $x^2 = -1$ has no solutions.

We can improve the situation by using $<$, which obeys the **order axioms**

- **O1, trichotomy:** For all $a, b \in \mathbb{R}$, precisely one of $a > b$, $a = b$ or $b < a$ is true;
- **O2, transitivity:** For all $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$ then $a < c$;
- **O3:** For all $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$;
- **O4:** For all $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$ then $ac < bc$.

The arithmetic and order axioms still don't distinguish between $\mathbb{R}$ and $\mathbb{Q}$ — so, for example, it is impossible to prove from them that $x^2 = 2$ has a solution, because this is true in $\mathbb{R}$ but not in $\mathbb{Q}$. The final axiom is:

The least upper bound property. Suppose that $S \subset \mathbb{R}$ has an upper bound. Then it has a least upper bound — that is, there exists an upper bound x for S such that if y is another upper bound for S then $y \geq x$.