ALGEBRA OF LIMITS

Proposition. *(Algebra of Limits, sequences) Suppose that $(a_n)_n$ and $(b_n)_n$ are sequences of reals such that $a_n \to a$ and $b_n \to b$. Then:*

(1) $a_n + b_n \to a + b$;
(2) $a_n b_n \to ab$;
(3) *if all b_n are non-zero* and $b \neq 0$, *then $a_n/b_n \to a/b$;*

Proof. (1) Let $\epsilon > 0$. As $a_n \to a$, there is $N_1 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for all $n \geq N_1$. As $b_n \to b$, there is $N_2 \in \mathbb{N}$ such that $|b_n - b| < \epsilon/2$ for all $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then, for all $n \geq N$, we have:

$$\begin{aligned} |a_n + b_n - (a+b)| &= |(a_n - a) + (b_n - b| \\ &\leq |(a_n - a)| + |(b_n - b)| && \text{(triangle inequality)} \\ &< \epsilon/2 + \epsilon/2 && \text{(as } n \geq N_1 \text{ and } n \geq N_2\text{)} \\ &= \epsilon \end{aligned}$$

as required.

(2) Let $\epsilon > 0$. First note that, since $(a_n)_n$ and $(b_n)_n$ are convergent, they are bounded; so pick X with $X \geq |a_n|$ and $X \geq |b_n|$ for all n, and $X \geq |a|, |b|$. Note that, for all n,

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)b_n + (b_n - b)a| && \text{(multiply out)} \\ &\leq |a_n - a||b_n| + |b_n - b||a|. \end{aligned}$$

As $a_n \to a$, there is $N_1 \in \mathbb{N}$ so that $|a_n - a| < \frac{\epsilon}{2X}$ for all $n \geq N_1$; similarly there is $N_2 \in \mathbb{N}$ so that $|b_n - b| < \frac{\epsilon}{2X}$ for all $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$ we have

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n - a||b_n| + |b_n - b||a| && \text{(above)} \\ &< \frac{\epsilon}{2X}(|b_n| + |a|) && \text{(as } n \geq N\text{)} \\ &\leq \frac{\epsilon}{2X}(X + X) && \text{(choice of } X\text{)} \\ &= \epsilon \end{aligned}$$

as required.

(3) We show that, if $b \neq 0$ and $b_n \neq 0$ for all n, then $1/b_n \to 1/b$. To get the full statement, then use part 2.

Let $\epsilon > 0$. Since $b \neq 0$ and $b_n \to b$, there is $M \in \mathbb{N}$ such that $|b_n - b| < b/2$ for $n \geq M$; so for $n \geq M$, $|b_n| \geq |b|/2$ (by the triangle inequality). There also exists $N_1 \in \mathbb{N}$ such that $|b_n - b| < \epsilon|b|^2/2$ for all $n \geq N_1$. Let

$N = \max(N_1, M)$. Then, for $n \geq N$, we have:

$$\begin{aligned}\left|\frac{1}{b_n} - \frac{1}{b}\right| &= \left|\frac{b - b_n}{bb_n}\right| \\ &= \frac{|b - b_n|}{|b||b_n|} \\ &< \frac{\epsilon|b|^2/2}{|b||b_n|} && \text{(as } n \geq N_1\text{)} \\ &\leq \frac{\epsilon|b|^2/2}{|b|^2/2} && \text{(as } n \geq M\text{)} \\ &= \epsilon.\end{aligned}$$

□

Proposition 0.1. *(Algebra of Limits, functions) Suppose that $A \subset \mathbb{R}$, a is a limit point of A, and that $f, g : A \to \mathbb{R}$ are functions such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. Then:*

(1) $\lim_{x\to a} f(x) + g(x) = l + m$;
(2) $\lim_{x\to a} f(x)g(x) = lm$;
(3) $\lim_{x\to a} \frac{f(x)}{g(x)} = l/m$ *if* $m \neq 0$ *and* $g(x) \neq 0$ *for all* $x \in A$.

Proof. This is basically identical to the proofs above, but using δ instead of N. In fact, I wrote this proof using copy and paste.

(1) Let $\epsilon > 0$. As $f(x) \to l$, there is $\delta_1 > 0$ such that $|f(x) - l| < \epsilon/2$ for all $x \in A$ with $0 < |x - a| < \delta_1$. As $g(x) \to m$, there is $\delta_2 > 0$ such that $|g(x) - m| < \epsilon/2$ for all $x \in A$ with $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2) > 0$. Then, for all $x \in A$ with $0 < |x - a| < \delta$, we have:

$$\begin{aligned}|f(x) + g(x) - (l + m)| &= |(f(x) - l) + (g(x) - m)| \\ &\leq |f(x) - l| + |g(x) - m| && \text{(triangle inequality)} \\ &< \epsilon/2 + \epsilon/2 && \text{(choice of } \delta\text{)} \\ &= \epsilon\end{aligned}$$

as required.

(2) Let $\epsilon > 0$. First note that, as $f(x) \to l$ and $g(x) \to m$ as $x \to a$, they are bounded near a; there is $X \in \mathbb{R}$ and $\delta' > 0$ such that $|f(x)| < X$ and $|g(x)| < X$ for $|x - a| < \delta'$. Note that, for all x,

$$\begin{aligned}|f(x)g(x) - lm| &= |(f(x) - l)g(x) + (g(x) - m)l| && \text{(multiply out)} \\ &\leq |f(x) - l||g(x)| + |g(x) - m||l|.\end{aligned}$$

As $f(x) \to l$, there is $\delta_1 > 0$ so that $|f(x) - l| < \frac{\epsilon}{2X}$ for all $x \in A$ with $0 < |x - a| < \delta_1$; similarly there is $\delta_2 > 0$ so that $|g(x) - m| < \frac{\epsilon}{2X}$ for all $x \in A$ with $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2, \delta') > 0$. Then for all $x \geq A$

with $0 < |x - a| < \delta$ we have

$$\begin{aligned} |f(x)g(x) - lm| &\leq |f(x) - l||g(x)| + |g(x) - m||l| && \text{(above)} \\ &< \frac{\epsilon}{2X}(|g(x)| + |l|) && \text{(as } 0 < |x - a| < \delta) \\ &\leq \frac{\epsilon}{2X}(X + X) && \text{(choice of } X \text{ and } \delta') \\ &= \epsilon \end{aligned}$$

as required.

(3) We show that, if $m \neq 0$ and $f(x) \neq 0$ for all n, then $1/f(x) \to 1/m$. To get the full statement, then use part 2.

Let $\epsilon > 0$. Since $m \neq 0$ and $f(x) \to m$, there is $\delta' > 0$ such that $|f(x) - m| < m/2$ for $0 < |x - a| < \delta'$; so for $0 < |x - a| < \delta'$, $|f(x)| \geq |m|/2$ (by the triangle inequality). There also exists $\delta_1 > 0$ such that $|f(x) - m| < \epsilon|m|^2/2$ for all $x \in A$ with $0 < |x - a| < \delta_1$. Let $\delta = \min(\delta_1, \delta') > 0$. Then, for $0 < |x - a| < \delta$, we have:

$$\begin{aligned} \left| \frac{1}{f(x)} - \frac{1}{m} \right| &= \left| \frac{m - f(x)}{mf(x)} \right| \\ &= \frac{|m - f(x)|}{|m||f(x)|} \\ &< \frac{\epsilon|m|^2/2}{|m||f(x)|} && \text{(as } 0 < |x - a| < \delta_1) \\ &\leq \frac{\epsilon|m|^2/2}{|m|^2/2} && \text{(as } 0 < |x - a| < \delta') \\ &= \epsilon. \end{aligned}$$

□