Multiple choice quiz

FOR EACH QUESTION, CHOOSE **ALL** CORRECT ANSWERS

1. WHICH OF THE FOLLOWING CAN BE PROVED FROM THE ARITHMETIC AXIOMS?

✓ (A) $(x+y)(x-y)=x^2-y^2$ [only difficulty is $(-y)x=-(yx)$]

(B) $1+1\neq 0$ $\left[\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \quad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}\right.$ is consistent with the axioms]

✓ (C) IF $ab=ac$ THEN $b=c$ OR $a=0$.

(D) IF $x^3=y^3$ THEN $x=y$. [for example, in $\mathbb{C}$ $\left(\frac{-1+i\sqrt{3}}{2}\right)^3=1$]

2. WHICH OF THE FOLLOWING CAN BE PROVED FROM THE ORDER AXIOMS + ARITHMETIC AXIOMS?

✓ (A) THERE EXISTS NO SOLUTION TO $x^2+1=0$. AS $1>0$, $x^2\geq 0$

(B) THERE EXISTS A SOLUTION TO $x^2-2=0$. $\sqrt{2}\in\mathbb{R}$, $\sqrt{2}\notin\mathbb{Q}$

(C) IF $x<\frac{1}{n}$ FOR ALL $n\in\mathbb{N}$, THEN $x\leq 0$. NEED ARCHIMEDEAN PRINCIPLE

(D) IF $x>y$ THEN $\frac{1}{x}<\frac{1}{y}$. e.g. $x=1$, $y=-1$.

3. WHICH OF THE FOLLOWING IDENTITIES FOR BINOMIAL COEFFICIENTS IS TRUE?

(A) $\binom{n}{k}+\binom{n}{n-1}=\binom{n+1}{k+1}$ if $1\leq k\leq n$

✓ (B) $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ if $1\leq k\leq n$

✓ (C) $\binom{n}{3}=\frac{n(n-1)(n-2)}{6}$ if $n\geq 3$

(D) $\binom{n}{k}=\binom{n+k}{n}$ if $0\leq k\leq n$. THE TRUE STATEMENT IS $\binom{n}{k}=\binom{n}{n-k}$

4. WHICH OF THE FOLLOWING IS ALWAYS TRUE FOR A NON-EMPTY BOUNDED-ABOVE SUBSET $S \subset \mathbb{R}$?

✓ (A) IF $x \geq y$ FOR ALL $y \in S$ THEN $x \geq \sup(S)$.

✓ (B) IF $x < \sup(S)$ ~~FOR~~ THEN $x < y$ FOR SOME $y \in S$

(C) $\sup(S) \in S$ eg $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ OR $S = \{x \in \mathbb{R} : x < 0\}$.

(D) IF $T = \{s^2 : s \in S\}$ THEN $\sup(T) = (\sup(S))^2$. eg $S = \{x \in \mathbb{R} : x < 0\}$

5. IN WHICH OF THE FOLLOWING CIRCUMSTANCES MUST A [redacted] FUNCTION f BE BOUNDED?

✓ (A) f IS A CONTINUOUS FUNCTION ON $(0, 1]$ AND $\lim_{x \to 0^+} f(x)$ EXISTS.

f EXTENDS TO CTS F^n ON $[0, 1]$

✓ (B) f IS A CONTINUOUS FUNCTION ON $[-7, 13]$.   EVT

✓ (C) f IS A CONTINUOUS FUNCTION ON $[0, \infty)$ AND $\lim_{x \to \infty} f(x)$ EXISTS

SEE MIDTERM 2.

✓ (D) g DEFINED BY $g(x) = f(x)^2$ IS A CONTINUOUS FUNCTION ON $[0, 1]$.   AS g IS BOUNDED, SO IS f.

6. WHICH OF THE FOLLOWING ARE CORRECT DEFINITIONS OF "$\lim_{x \to a} f(x) = L$"?

(A) THERE EXISTS $\delta > 0$ s.t. FOR ALL $\varepsilon > 0$, IF $0 < |x - a| < \delta$ THEN $|f(x) - L| < \varepsilon$.

(B) FOR EVERY $\varepsilon > 0$, THERE EXISTS $\delta > 0$ s.t. IF $|f(x) - L| < \delta$ THEN $|x - a| < \varepsilon$.

✓ (C) FOR EVERY $\varepsilon > 0$, THERE EXISTS $\delta > 0$ s.t. IF $0 < |x - a| < \delta$ THEN $|f(x) - L| < \varepsilon$.

✓ (D) FOR EVERY $\varepsilon > 0$, THERE EXISTS $\delta > 0$ s.t. IF $a - \delta < x < a + \delta$ AND $x \neq a$ THEN $L - \varepsilon < f(x) < L + \varepsilon$.

(7) WHICH OF THE FOLLOWING ARE TRUE?

(A) IF $\lim_{x \to a} (f(x)^2) = L^2$, THEN $\lim_{x \to a} f(x) = \pm L$. e.g. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$

✓ (B) IF $\lim_{x \to \infty} f(x) = L$, THEN $\lim_{x \to 0^+} f(\frac{1}{x}) = L$.

✓ (C) IF g IS CONTINUOUS AT L AND $\lim_{x \to a} f(x) = L$,

THEN $\lim_{x \to a} g(f(x)) = g(L)$.

✓ (D) $\lim_{x \to \infty} x \sin(\frac{1}{x}) = 1$

(8) WHICH OF THE FOLLOWING ARE CONTINUOUS AT 0?

✓ (A) $f(x) = \dfrac{x^3 + 2x - 5}{(x-7)^3}$

(B) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ $\lim_{x \to 0} f(x)$ DNE.

✓ (C) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

(D) $f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. $\lim_{x \to 0} f(x) = 0 \neq f(0)$.

(9) IF f, g ARE CONTINUOUS ON $\mathbb{R}$, WHICH OF THE FOLLOWING <u>MUST</u> ALSO BE CONTINUOUS ON <u>ALL</u> OF $\mathbb{R}$?

✓ (A) $f^2 + g$

(B) $\dfrac{f}{g}$ NOT IF $g(x) = 0$ FOR SOME x

✓ (C) $f \circ g$

(D) $\lfloor f \rfloor$ where $\lfloor f \rfloor(x) = \lfloor f(x) \rfloor = \max \{n \in \mathbb{Z} : n \leq x\}$.

10 WHICH OF THE FOLLOWING MUST HAVE A SOLUTION IN $[0,1]$?

(A) $f(x)=x$, WHERE f IS ANY CONTINUOUS FUNCTION. e.g. $f(x)=2$ $\forall x$

✓ (B) $x^3-7x+2=0$. $f(0)>0,\ f(1)<0$

✓ (C) $\cos(\pi x)=\pi x$. $\cos(0)>\pi\cdot 0$, $\cos(\pi)<\pi$

✓ (D) $x(1-x)f(x)=2x-1$, WHERE f IS ANY CONTINUOUS FUNCTION. AT 0, $LHS=0$, $RHS=-1$ AT 1, $LHS=0$, $RHS=1$

11 WHICH OF THE FOLLOWING FUNCTIONS ARE UNIFORMLY CONTINUOUS ON THE GIVEN INTERVALS ?

✓ (A) $\sin(x)$ ON $[0,\pi]$ BY THM

✓ (B) $\sin(x)$ ON $(0,\pi)$ IF YOU SHRINK THE DOMAIN IT'S STILL TRUE

(C) x^2 ON $[0,\infty)$

✓ (D) $\sqrt{x}$ ON $[0,\infty)$. SEE H/W.

12 WHICH OF THE FOLLOWING ARE DIFFERENTIABLE AT 0?

✓ (A) $f(x)=\begin{cases} 0 & \text{if } x=0 \\ x^2\sin(\frac{1}{x}) & \text{if } x\neq 0 \end{cases}$.

(B) $f(x)=|x|$.

✓ (C) $f(x)=x^3-7x+13\cos(\sin(x))$.

(D) $f(x)=\begin{cases} x & \text{if } x\in\mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$.

$$\frac{f(h)-f(0)}{h}=\begin{cases} 1 & h\in\mathbb{Q} \\ 0 & h\notin\mathbb{Q} \end{cases}$$

LIMIT DNE.

13 WHICH OF THE FOLLOWING IS TRUE FOR ANY DIFFERENTIABLE FUNCTIONS f & g ON $\mathbb{R}$?

(A) $(f \circ g)'(x) = f'(x)\, g'(f(x))$

✓ (B) $(f \circ g)'(x) = g'(x)\, f'(g(x))$ CHAIN RULE

(C) $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$ IF f IS INJECTIVE NOT IF $f'(f^{-1}(x)) = 0$

✓ (D) $(fg)'' = f''g + 2f'g' + fg''$. PRODUCT RULE TWICE.

14 IF f IS A DIFFERENTIABLE FUNCTION ON (a, b), WITH A MAXIMUM AT c, WHAT MUST BE TRUE?

✓ (A) $f'(c) = 0$

(B) $f''(c) < 0$

(C) $f''(c) \leq 0$

$f''(c)$ MIGHT NOT EXIST. IF IT DOES, (C) MUST BE TRUE

(D) THERE EXISTS $\delta > 0$ SUCH THAT f IS INCREASING ON $(c - \delta, c)$. e.g. $f(x) = x^2(1 - \sin(\frac{1}{x}))$, $c = 0$.

15 WHAT IS

$$\lim_{x \to 0} \left(\frac{\cos(x^2) - (\sin(x^2))^2 - 1}{x(x - \sin(x))} \right) ?$$

(A) 1

(B) π

(C) -5

✓ (D) -9

USE l'Hopital.

OR $\cos(t) \approx 1 - \dfrac{t^2}{2}$ as $t \to 0$ TAYLOR'S THEOREM

$\sin(t) \approx t - \dfrac{t^3}{6}$ as $t \to 0$

$\therefore$ TOP $\approx 1 - \dfrac{x^4}{2} - \left(x^2 - \dfrac{x^6}{6}\right)^2 - 1$

$\approx \dfrac{-3x^4}{2}$

BOTTOM $\approx x \cdot \left(x - \left(x - \dfrac{x^3}{6}\right)\right)$

$\approx \dfrac{x^4}{6}$

$\ell\ \dfrac{-3x^4/2}{x^4/6} = \underline{\underline{-9}}$