a) $\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$

b) For $\lambda \in \mathbb{R}$,

$$\underbrace{\langle v - \lambda w, v - \lambda w \rangle}_{= |v - \lambda w|^2} \geq 0 .$$

So $\langle v, v \rangle - \lambda \langle w, v \rangle - \lambda \langle v, w \rangle + \lambda^2 \langle w, w \rangle \geq 0 \quad \forall \lambda \in \mathbb{R}$

$\therefore \quad A - B\lambda + C\lambda^2 \geq 0 \quad \forall \lambda \in \mathbb{R}$ where $A = \langle v, v \rangle$
$B = 2\langle v, w \rangle$
$C = \langle w, w \rangle$.

This is only possible if $B^2 - 4AC \leq 0$, because of the quadratic formula.

$$\therefore \quad (2\langle v, w \rangle)^2 - 4\langle v, v \rangle \langle w, w \rangle \leq 0$$

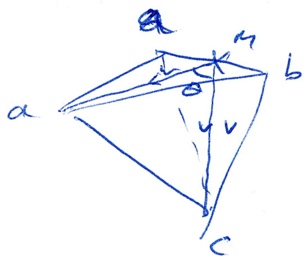
$$\therefore \quad \cancel{4}\langle v, w \rangle^2 \leq \cancel{4}\, |v|^2 |w|^2$$

$$\therefore \quad \boxed{\langle v, w \rangle \leq |v|\,|w|}$$

this is the Cauchy-Schwarz inequality.

c) m has coordinates $\frac{1}{2}(a+b) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$.

Let $v = c - m = \left(\frac{-1}{2}, \frac{-1}{2}, 1, 0\right)$
$w = d - m = \left(\frac{-1}{2}, \frac{-1}{2}, 0, 1\right)$.



Then $|v|\,|w| \cos\theta = \langle v, w \rangle$

$$\therefore \quad \boxed{\cos\theta = \frac{\langle v, w \rangle}{|v|\,|w|} = \frac{\frac{1}{4} + \frac{1}{4} + 0 + 0}{\sqrt{\frac{3}{2}} \cdot \sqrt{\frac{3}{2}}} = \frac{1/2}{3/2} = \frac{1}{3}}$$

(2a) $U \subset \mathbb{R}^n$ is open if, $\forall x \in U$, $\exists \varepsilon > 0$

such that if $|y-x| < \varepsilon$ then $y \in U$.

(bi) True: let $x \in U \cap V$.

Then $x \in U$, so $\exists \varepsilon_1 > 0$ s.t. $|y-x| < \varepsilon_1 \Rightarrow y \in U_\varepsilon$

& $x \in V$ so $\exists \varepsilon_2 > 0$ s.t. $|y-x| < \varepsilon_2 \Rightarrow y \in V$.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then

$$|y-x| < \varepsilon \Rightarrow |y-x| < \varepsilon_1 \ \& \ |y-x| < \varepsilon_2$$

$$\Rightarrow x \in U \ \& \ x \in V$$

$$\Rightarrow x \in U \cap V.$$

So $U \cap V$ open ✓.

(ii) false take $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$, $U = \mathbb{R}$ open, $f(U) = [0, \infty)$ not open.

(iii) false take $X_n = [\frac{1}{n}, \infty)$ closed

$\bigcup_{n=1}^{\infty} X_n = (0, \infty)$ not closed.

(iv) true let $x \in f^{-1}(U)$, so $f(x) = y \in U$.

$\exists \varepsilon > 0$ s.t. if $|y'-y| < \varepsilon$ then $y' \in U$, as U open. (1)

$\exists \delta > 0$ s.t. if $|x'-x| < \delta$ then $|f(x')-f(x)| < \varepsilon$, as f cts (2)

$$\therefore \quad |x'-x| < \delta \Rightarrow |f(x') - \overset{\text{"}y\text{"}}{f(x)}| < \varepsilon \quad \text{by (2)}$$

$$\Rightarrow f(x') \in U \quad \text{by (1)}$$

$$\Rightarrow x' \in f^{-1}(U).$$

So $f^{-1}(U)$ is open.

(3a). Let $X \subset \mathbb{R}^n$ compact.

For $r > 0$, let

$$U_r = \{x \in X : |x| < r\}.$$

It is open, and

$\mathcal{C} = \{U_r : r > 0\}$ is an open cover of X

(as if $x \in X$, then $x \in U_r$ for $r > |x|$).

~~But if $U_{r_1}, \ldots, U_{r_n}$~~

As X is compact, there is a finite subcover

$$X = U_{r_1} \cup \ldots \cup U_{r_k}, \quad r_1, \ldots, r_k \geq 0.$$

Let $r = \max(U_{r_1}, \ldots, U_{r_k})$.

Then $x \in X \Rightarrow x \in U_{r_i}$ some $i = 1, \ldots, k$

$$\Rightarrow |x| < r_i \leq r.$$

So $|x| < r \;\; \forall x \in X$ i.e. X bounded.

(b) Since ~~[illegible]~~ T is continuous (any linear map is)

& 1-1 is continuous, f is continuous for $x \neq 0$.

The set $\{x \in \mathbb{R}^n : |x| = 1\} = S$ is closed & bounded

$\therefore$ f has a maximum value on S by the EVT; i.e.

$\exists\, x_0 \in S$ s.t. $f(x) \leq f(x_0) \;\; \forall x \in S$, But now if $x \in \mathbb{R}^n \setminus \{\underline{0}\}$, then $\hat{x} = \dfrac{x}{|x|} \in S$.

Then

$$f(x) = \frac{T(x)}{|x|} = \frac{T(|x|\hat{x})}{|x|} \overset{\text{Then}}{\underset{\downarrow}{=}} \frac{|x|\,T(\hat{x})}{|x|}$$

$$= \frac{T(\hat{x})}{\cancel{\ \ }} \underset{\substack{\uparrow \\ |\hat{x}|=1}}{=} \frac{T(\hat{x})}{|\hat{x}|} = f(\hat{x}) \leq f(x_0).$$

So $f(x) \leq f(x_0) \quad \forall x \in \mathbb{R}^n \setminus \{\underline{0}\}$.

$\therefore$ $f(x_0)$ is max. value of f on all of $\mathbb{R}^n \setminus \{\underline{0}\}$.

4a. $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - T(h)}{|h|} = 0.$$

b) ~~$f((x+h)(y+k)(z+l)) =$~~

$$f(x+h, y+k, z+l) = (x+h)(y+k)(z+l)$$

$$= \underbrace{xyz}_{f(x,y,z)} + \underbrace{yzh + zxk + xyl}_{T(h,k,l)} + xkl + yhl + zhk + hkl.$$

Define $T(h,k,l) = yzh + zxk + xyl$

Then

$$\frac{|f(x+h,\ y+k,\ z+l) - f(xyz) - T(h,k,l)|}{\sqrt{h^2+k^2+l^2}}$$

$$= \frac{|xkl + yhl + zhk + hkl|}{|\sqrt{\quad}|}$$

$$= \left| \frac{h}{\sqrt{h^2+k^2+l^2}} (yl + zk + kl) + \frac{k}{\sqrt{h^2+k^2+l^2}} (xl) \right|$$

$$\leq \underbrace{\left| \frac{h}{\sqrt{\cdots}} \right|}_{\leq 1} |yl + zk + kl| + \underbrace{\left| \frac{k}{\sqrt{\cdots}} \right|}_{\leq 1} |xl|$$

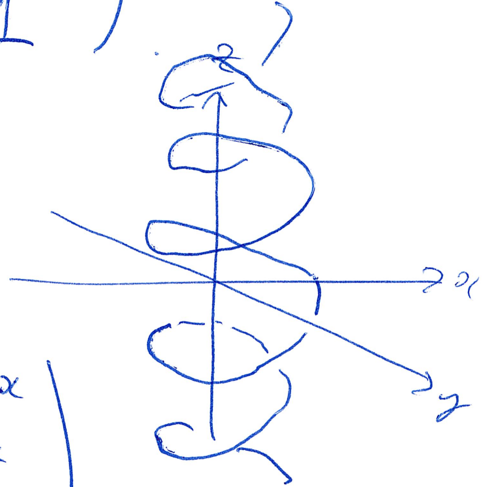
$$\leq |yl + zk + kl| + |xl| \longrightarrow 0 \text{ as } (h,k,l) \to (0,0,0)$$

So $Df(x,y,z) = T$ $\quad\blacksquare$.

(So $f'(x,y,z) = \begin{pmatrix} yz & zx & xy \end{pmatrix}$).

(c) $$g'(x) = \begin{pmatrix} g_1'(x) \\ g_2'(x) \\ g_3'(x) \end{pmatrix} = \begin{pmatrix} -\sin x \\ \cos x \\ 1 \end{pmatrix}$$

(d)



$(f \circ g)'(x)$

$$= f'(g(x))\, g'(x)$$

$$= \begin{pmatrix} x\sin x & x\cos x & \cos x \sin x \end{pmatrix} \begin{pmatrix} -\sin x \\ \cos x \\ 1 \end{pmatrix}$$

$$= -x\sin^2 x + x\cos^2 x + \cos x \sin x = \underline{\underline{x(\cos^2 x - \sin^2 x) + \cos x \sin x}}.$$