HW9 - solutions

$x=0$

1i) $f(x)=e^{e^x}$ $\qquad e$

$f'(x)=e^x e^{e^x}$ $\qquad e$

$f''(x)=e^x e^{e^x}+e^{2x}e^{e^x}$ $\qquad 2e$

$f'''(x)=e^x e^{e^x}+3e^{2x}e^{e^x}+e^{3x}e^{e^x}$ $\qquad 5e$

$$\to P_{3,0}(x)=e+ex+\frac{2ex^2}{2!}+\frac{5}{3!}ex^3$$

$$=e\left(1+x+x^2+\frac{5}{6}x^3\right).$$

iii) $\sin^{(k)}\left(\frac{\pi}{2}\right)=\begin{cases}(-1)^j & \text{if } k=2j \text{ is even}\\ 0 & \text{if } k \text{ is odd}\end{cases}$

$$\therefore P_{2n,\frac{\pi}{2}}(x)=1-\frac{(x-\frac{\pi}{2})^2}{2!}+\frac{(x-\frac{\pi}{2})^4}{4!}-\dots+(-1)^n\frac{(x-\frac{\pi}{2})^{2n}}{2n!}.$$

(alternative solution: $\sin\left(\frac{\pi}{2}+x\right)=\cos(x)$, so

$P_{2n,\frac{\pi}{2}}(x)=Q_{2n,0}\left(x-\frac{\pi}{2}\right)$ where Q is the Taylor polynomial of cos).

v) $f^{(k)}(1)=e$ for all k

$$\to P_{n,1}(x)=e\left(1+(x-1)+\frac{(x-1)^2}{2!}+\dots+\frac{(x-1)^n}{n!}\right).$$

vii) $x+x^3$

viii) $f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{6}(x-1)^3+\frac{f^{(4)}(1)}{24}(x-1)^4$

$$=3+9(x-1)+13(x-1)^2+11(x-1)^3+5(x-1)^4.$$

2iv) $(9a+3b+c) + (6a+b)(x-3) + a(x-3)^2$

3i) $= f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2.$

3ii) The error term $R_{2n+1,0}(x)$ is $\frac{f^{(2n+2)}(t)}{(2n+2)!} \cdot 2^{2n+2}$ for some $t \in (0,2)$.

$$\therefore |R_{2n+1,0}(2)| = \frac{|\sin(t)|}{(2n+2)!} \cdot 2^{2n+2}$$

$$\leq \frac{2^{2n+2}}{(2n+2)!}.$$

Looking at the table, $\frac{2^{20}}{20!} < \frac{2\times 10^6}{2\times 10^{18}} = 10^{-12}$

$\therefore \sin(2) = \sum_{n=0}^{9} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}$ to within 10^{-12}.

4ii) The error term in the Taylor expansion $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + R_{n,0}(1)$ is

$R_{n,0}(1) = \frac{e^t}{(n+1)!} \cdot 1^n$ for some $t \in (0,1)$; as $e < 3$,

$|R_{n,0}(1)| < \frac{3}{(n+1)!}$ and we must find n such that $(n+1)! > 3 \cdot 10^{1000}$.

if $n > 10$, $(n+1)! > 9! \cdot 10^{n+1-9} > 3 \cdot 10^{n-8}$

$\therefore n = 1008$ will work (although this is very far from optimal!)

$$\therefore \left| e - \sum_{n=0}^{1008} \frac{1}{n!} \right| < 10^{-1000}.$$

6a) By 15-9,

$$\arctan(x)+\arctan(y)=\arctan\left(\frac{x+y}{1-xy}\right). \quad (*)$$

$$\therefore \arctan\left(\tfrac{1}{2}\right)+\arctan\left(\tfrac{1}{3}\right)=\arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$=\arctan(1) \;=\; \pi/4 \quad ✓$$

b) $\arctan\left(\tfrac{1}{5}\right)+\arctan\left(\tfrac{1}{5}\right)=\arctan\left(\tfrac{5}{12}\right)$ by $(*)$

$\arctan\left(\tfrac{5}{12}\right)+\arctan\left(\tfrac{5}{12}\right)=\arctan\left(\tfrac{120}{119}\right)$ by $(*)$

$\arctan(1)+\arctan\left(\tfrac{1}{239}\right)=\arctan\left(\tfrac{240}{238}\right)$ by $(*)$

$=\arctan\left(\tfrac{120}{119}\right)$

$$\therefore \arctan(1)+\arctan\left(\tfrac{1}{239}\right)=4\arctan\left(\tfrac{1}{5}\right)$$

$$\therefore \frac{\pi}{4}=4\arctan\left(\tfrac{1}{5}\right)-\arctan\left(\tfrac{1}{239}\right)$$

Now use Taylor series for $\arctan\left(\frac{1}{5}\right)$, $\arctan\left(\frac{1}{239}\right)$; get

$$\arctan(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots+\frac{(-1)^n x^{2n+1}}{2n+1}+R_n(x)$$

where $\quad |R_n(x)|\leq\dfrac{|x|^{2n+3}}{2n+3}.$

$$\therefore \quad \pi\approx 16\left(\frac{1}{5}-\frac{1}{3\cdot125}+\frac{1}{5\cdot5^5}-\frac{1}{7\cdot5^7}+\frac{1}{9\cdot5^9}\right)-4\cdot\frac{1}{239}$$

$$=3.141592\dots$$

with error ~~is~~

$$\leq 16\cdot\frac{1}{11\cdot5^{11}}+\frac{4}{3\cdot239^3}<10^{-6}$$

$\Rightarrow$ first 6 digits of π are 3.14159.

7 i) since $\cancel{f^{(k)}(a)}$ $(f+g)^{(k)}(a) = f^{(k)}(a) + g^{(k)}(a)$,

the nth Taylor polynomial $\sum_{k=0}^{n} c_k (x-a)^k$ has

$$c_k = a_k + b_k .$$

ii) $(f \cdot g)^{(k)}(a) = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(a) g^{(k-i)}(a)$ & $\binom{k}{i} = \frac{k!}{i!(k-i)!}$

$$\therefore \quad \frac{(f \cdot g)^{(k)}(a)}{k!} = \sum_{i=0}^{k} \overbrace{\frac{f^{(i)}(a)}{i!}}^{a_i} \overbrace{\frac{g^{(k-i)}(a)}{(k-i)!}}^{b_{k-i}}$$

$$\therefore \quad c_k = a_0 b_k + a_1 b_{k-1} + \dots + a_{k-1} b_1 + a_k b_0 = \sum_{i=0}^{k} a_i b_{k-i}$$

iii) as $(f')^{(k)} = f^{(k+1)}$,

$$c_k = \frac{f'^{(k)}(a)}{k!} = \frac{f^{(k+1)}(a)}{k!} = (k+1) a_k .$$

iv) $h^{(k)} = f^{(k-1)}$ if $k \geq 1$, by FTC

& $h^{(0)}(a) = 0$

$$\therefore \quad c_k = \begin{cases} 0 & \text{if } k = 0 \\ \frac{a_k}{k} & \text{if } k \geq 1 \end{cases}$$

8a) Let ~~the~~

$$P(x) = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Have to show $P(x^2)$ is the degree $4n+2$ Taylor poly of $\sin(x^2)$ at 0.

~~ie~~ So suffices to show

$$\frac{\sin(x^2) - P(x^2)}{x^{4n+2}} \to 0 \text{ as } x \to 0$$

, as the Taylor polynomial is determined by this property.

But $$\frac{\sin(x) - P(x)}{x^{2n+1}} \to 0 \text{ as } x \to 0$$

, as P is the $(2n+1)$st Taylor poly for $\sin$.

Replacing x by x^2, $$\frac{\sin(x^2) - P(x^2)}{x^{4n+2}} \to 0 \text{ as } x \to 0$$ as required.

12) $\sin(x) = x - \frac{x^3}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \tilde{R}_{2n+1,0}(x)$ where

$$|\tilde{R}_{2n+1,0}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}.$$

$\therefore$ $$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n,0,f}(x)$$

where $|R_{2n,0,f}(x)| \leq \frac{|x|^{2n+1}}{(2n+2)!}$. ~~E~~

(if $x = 0$, $1 = 1$ & $R_{2n,0,f}(0) = 0$).

$\therefore$ $$\int_0^1 f = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \left(\text{error} < \frac{x^5}{6!}\right)\right)$$ $6! = 720$.

$$= 1 - \frac{1}{18} + \frac{1}{600} \quad \cancel{+ \left(\text{error} < \frac{1}{6!} <\right.}$$

$$+ \left(\text{error} < \int_0^1 \frac{x^5}{6!} = \frac{1}{6 \cdot 6!} < 10^{-3}\right)$$

$$= 0.946\dots$$

15) $R_{n,0}(x) = \frac{e^t}{(n+1)!} x^{n+1}$ for $t \in (x, 0)$, if $x < 0$.

$\therefore \ |R_{n,0}(x)| = \frac{|x|^{n+1}}{(n+1)!} e^t$

$\le \frac{|x|^{n+1}}{(n+1)!}$ as $t < 0$.

16) ~~$R_{n,0}(x)$~~

As $\log(1+x) = \int_0^{x} \frac{1}{1+x} \, dx$

$= \int_0^x 1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t} \, dt$

$= x - \frac{x^2}{2} + - \quad + \frac{(-1)^n x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t} \, dt$,

see $|R_{n,0}(x)| = \left| \int_0^x \frac{t^n}{1+t} \, dt \right|$.

If $x < 0$, ~~this is $\le |x|$.~~

then $1+t > 1+x$ for $t \in (x, 0)$

$\therefore \ \frac{1}{1+t} < \frac{1}{1+x}$. $\quad \therefore \ |R_{n,0}(x)| \le \frac{1}{1+x} \left| \int_0^x t^n \, dt \right|$

$= \frac{|x|^{n+1}}{(1+x)(n+1)}$. $\square$

(25) Suppose $f''=f$, $f(0)=f'(0)=0$.

Then $f^{(n)}(0)=0$ for all n.

For any x,

$$\sup\{f^{(n)}(t) : t\in[0,x]\} = \begin{cases} \sup\{f(t): t\in[0,x]\} & n \text{ even} \\ \sup\{f'(t): t\in[0,x]\} & n \text{ odd} \end{cases}$$

$$\leq C$$

for some C independent of n (as f, f' are cts so bounded on $[0,x]$).

$$\therefore \quad f(x) = 0 + 0x + \frac{0x^2}{2} + \frac{0x^3}{3!} + \dots + \frac{0x^n}{n!} + R_{n,0}(x)$$

where
$$|R_{n,0}(x)| \leq \frac{C|x|^n}{n!} \to 0 \quad \text{as } n\to\infty.$$

$$\therefore \quad \boxed{f(x)=0 \quad \text{for all } x}$$

By same argument as for $f''+f=0$, see general solution to $f''=f$ is

$$f(x) = f(0)\cosh(x) + f'(0)\sinh(x)$$

where $\cosh(x) = \dfrac{e^x+e^{-x}}{2}$, $\sinh(x) = \dfrac{e^x-e^{-x}}{2}$.